25.07.2018
Arithmatic
Seminar
Talk #1

Uniform Distribution (mod 1)

We consider $\mathbb{R}/\mathbb{Z}$ $\left(= [0,1) \right)$

Let $X = (x_n)_{n\in\mathbb{N}} \in \left(\mathbb{R}/\mathbb{Z}\right)^{\mathbb{N}}$

For each $0 \leq a < b < 1$ define φ_N as follow:

$$\varphi_N(a,b) = \left| \left\{ n \leq N : \ x_n \in [a,b) \right\} \right|$$

<u>Definition</u>: If $\lim\limits_{N\to\infty} \frac{1}{N}\varphi_N(a,b) = b-a$, then X is said to be a uniformly distributed sequence.

<u>Remark</u>: (x_n) is uniformly distributed, $\alpha \in \mathbb{R}/\mathbb{Z}$
$\Rightarrow$ $(x_n+\alpha)$ is uniformly distributed (mod 1).

<u>Remark</u>: $X = (x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ is uniformly distributed modulo 1 $:\Leftrightarrow$ $\left(x_n - [x_n]\right)_{n\in\mathbb{N}} = \left(\{x_n\}\right)_{n\in\mathbb{N}}$ is uniformly distributed.

<u>Remark</u>: $X = (x_n)_{n\in\mathbb{N}}$ is uniformly distributed modulo 1 $\Rightarrow$ $\{x_n : \ n\in\mathbb{N}\}$ is dense in $[0,1)$.

Remark: $\left(\{\log(n)\}\right)_{n\in\mathbb{N}}$ is dense in $[0,1)$

But it is not uniformly distributed.

Theorem: (H. Weyl)

Let $X=(x_n)_{n\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}$. Then X is uniformly distributed modulo 1 iff $\forall k\in\mathbb{Z}^{*}$, we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i k x_n}=0$$

Example: Let $\alpha\in\mathbb{R}\setminus\mathbb{Q}$. Consider $x_n=\alpha n$

$$\sum_{n=1}^{N}e^{2\pi i k\alpha n}=\frac{e\left((N+1)\alpha\right)-e(k\alpha)}{1-e(k\alpha)};$$

$$e(z)=e^{2\pi i z}$$

$$\Rightarrow\left|\frac{1}{N}\sum_{n=1}^{N}e^{2k\pi i\alpha n}\right|\leqslant\frac{1}{N}\frac{2}{|1-e(k\alpha)|}\longrightarrow 0$$

as $N\to\infty$.

$\Rightarrow(n\alpha)_{n\in\mathbb{N}}$ is uniformly distributed modulo 1.

Example: The sequence $(\log n)_{n\in\mathbb{N}}$ is not uniformly distributed

Sketch of proof:

Recall:

$f: \mathbb{R} \longrightarrow \mathbb{R}$, $f \in C^1$, $(a_n) \in \mathbb{R}^{\mathbb{N}} \Rightarrow$

$$A(t) = \sum_{n \leq t} a_n$$

$$\sum_{1\leq n\leq x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)\,dt$$

$$f(t) = e^{2\pi i \log t} = t^{2\pi i}, \quad a_n = 1$$

$$\sum_{1\leq n\leq N} e(\log n) = \cdots = \frac{N^{1+2\pi i}}{1+2\pi i} + O(\log N) \neq o(N)$$

Lemma: Let $X \in (\mathbb{R}/\mathbb{Z})^{\mathbb{N}}$. Then TFAE:

(a) X is uniformly distributed modulo 1.

(b) $\forall f \in C(\mathbb{R}/\mathbb{Z})$ $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)\,dx$

$$\left(\varphi_N(I) = \left| \{ n \leq N : x_n \in I \} \right| = \sum_{n=1}^{N} \chi_I(x_n) \right)$$

(c) $\forall f \in \mathcal{R}(\mathbb{R}/\mathbb{Z})$, $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)\,dx$

proof: (c) $\Rightarrow$ (a) is trivial (Just put $f = \chi_I$)

(a) $\Rightarrow$ (c): X is uniformly distributed $\Rightarrow$

(*) holds true for any step function s

$$s(x) = \sum a_n \chi_{I_n}(x).$$

Now, let $f \in \mathcal{R}(\mathbb{R}/\mathbb{Z})$, $\varepsilon > 0$. There exist step functions on $[0,1)$ s.t. $s \leq f \leq t$ and

$$\int_0^1 (t(x) - s(x))\,dx < \varepsilon.$$

For t, $\exists N(\varepsilon) \in \mathbb{N}$ $\forall N \geq N(\varepsilon)$

$$\left| \frac{1}{N} \sum_{n=1}^{N} t(x_n) - \int_0^1 t(x)\,dx \right| < \varepsilon$$

$$\frac{1}{N} \sum_{n=1}^{N} f(x_n) - \int_0^1 f(x)\,dx = \underbrace{\frac{1}{N} \left(\sum_{n=1}^{N} f(x_n) - t(x_n) \right)}_{\leq 0} + \frac{1}{N} \sum_{n=1}^{N} t(x_n)$$

$$- \int_0^1 t(x)\,dx + \int_0^1 t(x)\,dx - \int_0^1 f(x)\,dx \leq \left(\frac{1}{N} \sum_{n=1}^{N} t(x_n) - \int_0^1 t(x)\,dx \right)$$

$$+ \left(\int_0^1 (t(x) - f(x))\,dx \right) < 2\varepsilon.$$

similarly, by using $s(x)$ instead of $t(x)$, we get

$$\frac{1}{N}\sum_{n=1}^{N} f(x_n) - \int_0^1 f(x)dx > -2\varepsilon$$

$$\Rightarrow \left| \frac{1}{N}\sum_{n=1}^{N} f(x_n) - \int_0^1 f(x)dx \right| < 2\varepsilon \Rightarrow \text{the result follows.}$$

(c) $\Rightarrow$ (b): ✓

(b) $\Rightarrow$ (a): follows from approximating the characteristic functions of intervals by continuous functions from above and below.

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proof of the Weyl criterion:

X is uniformly distributed iff $\forall k \in \mathbb{Z}^*$

$$\frac{1}{N}\sum_{n=1}^{N} e(kx_n) \longrightarrow 0 \quad \text{as } N \to \infty \qquad (**)$$

($\Rightarrow$) follows from the lemma above.

($\Leftarrow$) (**) implies that for every trigonometric polynomial P, $\frac{1}{N}\sum_{n=1}^{N} P(x_n) \longrightarrow 0$ as $N \to \infty$

Let $f \in C(\mathbb{R}/\mathbb{Z})$. By Stone-Weierstrass Theorem, we can find a trigonometric polynomial T s.t. $\|f - T\|_\infty < \varepsilon$

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(x_n) - \int_0^1 f(x)\,dx \right| \leqslant \left| \frac{1}{N} \sum_{n=1}^{N} \big(f(x_n) - T(x_n)\big) \right|$$

$$+ \left| \frac{1}{N} \sum_{n=1}^{N} T(x_n) - \int_0^1 T(x)\,dx \right| + \left| \int_0^1 T(x)\,dx - \int_0^1 f(x)\,dx \right| < 3\varepsilon$$

for N large enough. ∎

Some Variants & Generalizations

Let $Q \subseteq \mathbb{R}^p$ be the unit cube, $X \in Q^{\mathbb{N}}$. If $V \subseteq Q$ is another cube,

$$\varphi_N(V) = \left| \{ n \leqslant N : \; x(n) \in V \} \right|$$

$$\forall V \subseteq Q, \quad \lim_{N \to \infty} \frac{\varphi_N(V)}{N} = m(V)$$

we say that X is uniformly distributed.

Theorem: X is uniformly distributed iff

$$\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i (k \cdot x(n))} \longrightarrow 0 \text{ as } N \to \infty \quad \forall k \in \mathbb{Z}^p - \{(0, \dots, 0)\}.$$

Conclusion: $\alpha_1, \dots, \alpha_p \in \mathbb{R}$, $\{\alpha_1, \dots, \alpha_p, 1\}$ is linearly independent over $\mathbb{Z}$ $\Rightarrow$ $\big(n\alpha = (n\alpha_1, \dots, n\alpha_p)\big)_{n \in \mathbb{N}}$ is uniformly distributed mod Q

Another view:

$\alpha, \beta \in \mathbb{R}$, $a_n = \alpha n + \beta$

$T_\alpha : x \longrightarrow x + \alpha \Rightarrow$ Then there is only one measure μ on S^1 s.t. $\mu(A) = \mu(T_\alpha^{-1}(A))$ for all meaningful A.

Theorem: (Weyl)

$\alpha \notin \mathbb{Q} \Rightarrow (n^2\alpha)_{n\in\mathbb{N}}$ is uniformly distributed.

Theorem: $P(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0$ is a polynomial with $d \geqslant 2$, $a_d \neq 0$. If at least one of $a_1, \cdots, a_d$ is irrational, then $(P(n))_{n\in\mathbb{N}}$ is uniformly distributed.

Theorem: Let $(a(n))_{n\in\mathbb{N}}$ be a sequence of distinct integers, i.e., $a(n) \neq a(m)$ when $n \neq m$.

Then, for almost all $\alpha \in \mathbb{R}$, $(a(n)\alpha)_{n\in\mathbb{N}}$ is uniformly distributed.

<u>Definition</u>: $\alpha \in (0,1)$ is said to be normal iff for any $t \in \{0, 1, \dots, 9\}$

$$\lim_{N\to\infty} \frac{1}{N} \left| \{ n \leq N : a_n = t \} \right| = \frac{1}{10}$$

, where $\alpha = 0, a_1 a_2 a_3 \dots a_n \dots$

<u>Examples</u>:

1. Champernowne ('33): $0,12345678910111213\dots$
2. Copeland - Erdös ('46): $0,2357111319232 9\dots$
3. Pavenpart - Erdös ('52): $f(x) \in \mathbb{R}[x]$ s.t. $f(n) \in \mathbb{N}$ $\forall n \in \mathbb{N} \Rightarrow 0, f(1) f(2) f(3) \dots$ is normal.

<u>Question</u>: Let $f(x) \in \mathbb{R}[x]$ s.t. $f(n) \in \mathbb{N}$ $\forall n \in \mathbb{N}$.
Let p_i be ith prime.
Is the number $0, f(p_1) f(p_2) f(p_3) \dots$ normal?

<u>Theorem</u>: $\alpha \in (0,1)$ is normal iff $(10^n \alpha)_{n \in \mathbb{N}}$ is uniformly distributed.

<u>Question</u>: Find an irrational number which is not normal. (<u>Answer</u>: $0,01001000100001\dots$)

<u>Problem</u>: Find more normal normal numbers!
(Are $\pi, e, \sqrt{3}, \dots$ normal?)

References:

1. Equidistribution in Number Theory, A. Granfille

2. Uniform Distribution of Sequences - Kuipers.