05.10.2018
Arithmetic
Seminar
Talk - 4

Birkhoff Recurrence Theorem

Let X be a compact metric space, $T: X \longrightarrow X$ be a continuous function. Then there is a point $x \in X$ and a sequence $n_1 < n_2 < \cdots$ such that $T^{n_k}(x) \longrightarrow x$ as $k \longrightarrow \infty$.

B. Multiple Recurrence Theorem (MBR)

Let X and T be above. $\forall p \geq 1 \ \exists x \in X \ \exists n_k \nearrow \infty$ such that

$$T^{n_k}(x) \longrightarrow x$$

$$(T^2)^{n_k}(x) \longrightarrow x$$

$$\vdots$$

$$(T^p)^{n_k}(x) \longrightarrow x$$

Proof of Van der Waerden Theorem.

Let $\Lambda = \{1, \ldots, r\}$ be the set of colors. Let $\Omega = \Lambda^{\mathbb{Z}}$ be the space of all colorings of $\mathbb{Z}$.

$x, y \in \Omega$ & $x \neq y \Rightarrow d(x,y) := 2^{-\min\{\ell \geq 0 : \, x(\ell) \neq y(\ell) \text{ or } x(-\ell) \neq y(-\ell)\}}$

suppose $d(x,x) = 0$.

1. (Ω, d) is a compact metric space.
2. $\varepsilon < 1/2^m$ and $d(x,y) < \varepsilon \Rightarrow x(n) = y(n) \ \forall n \in [-m, m]$

Define $T: \Omega \longrightarrow \Omega$ by $T(x)(n) = x(n+1)$. Let $\xi \in \Omega$ be our coloring.

$X = \overline{\{T^n(\xi) : n \in \mathbb{Z}\}} \Rightarrow X$ is compact & T-invariant

subset of Ω $\left(T(X) \subseteq X \right)$. $T|_X : X \longrightarrow X$.
By the BMR Theorem, for $p \geq 1$ in the hypothesis in the VDW Theorem, $\exists x \in X, \exists n_k \nearrow \infty$

$$T^{n_k}(x) \longrightarrow x, \ (T^2)^{n_k}(x) \longrightarrow x, \ \dots, (T^p)^{n_k}(x) \longrightarrow x.$$

$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} :$

$$d\left(T^N(x), x\right) < \varepsilon$$

$$d\left((T^2)^N(x), x\right) < \varepsilon$$

$$\vdots$$

$$d\left((T^p)^N(x), x\right) < \varepsilon$$

$$\varepsilon \leq 1/2 \Rightarrow x(0) = T^N(x)(0) = x(N) = x(2N) = x(3N) \dots = x(pN)$$

By the definition of X, $\exists m \in \mathbb{Z}$:

$$d\left(T^m(\xi), x\right) < \frac{1}{2^{pN+1}}$$

$$T^m(\xi)(0) = T^m(\xi)(N) = \dots = T^m(\xi)(pN)$$

$$\xi(m) = \xi(m+N) = \xi(m+2N) = \dots = \xi(m+pN)$$

$\blacksquare$

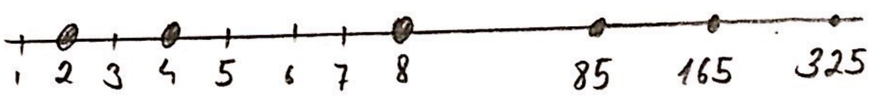
Theorem 1. $\forall k, r \geq 1$, $\exists N = N(k,r)$: whenever you color $[1, N]$ with r colors you will find a monochromatic k-term AP.

Theorem 2. $\forall k \geq 1$ $\forall r \geq 1$, whenever you color $\mathbb{N}$ with r colors, you will find a monochromatic k-term AP.

proof of Theorem 1.
Suppose Theorem 1. fails to be true. $\exists k, r \geq 1$: $\forall N \in \mathbb{N}$ there is an r-coloring $f_N : [1,N] \longrightarrow \Lambda$ there is no k-term AP.
Extend each f_N to $\mathbb{Z}$ arbitrarily and get g_N.
Since Ω is compact, there is a subsequence $g_{n_k} \longrightarrow g \in \Omega \Rightarrow g$ does not have any k-term A.P., a contradiction.

Conclusion. Suppose we color elements of $\mathbb{Z}$ by using r colors. $\forall F \subseteq \mathbb{Z}$ finite : $\exists a, b \in \mathbb{Z}$:

$aF + b \subseteq B$ where B is a 1-color part of $\mathbb{Z}$.

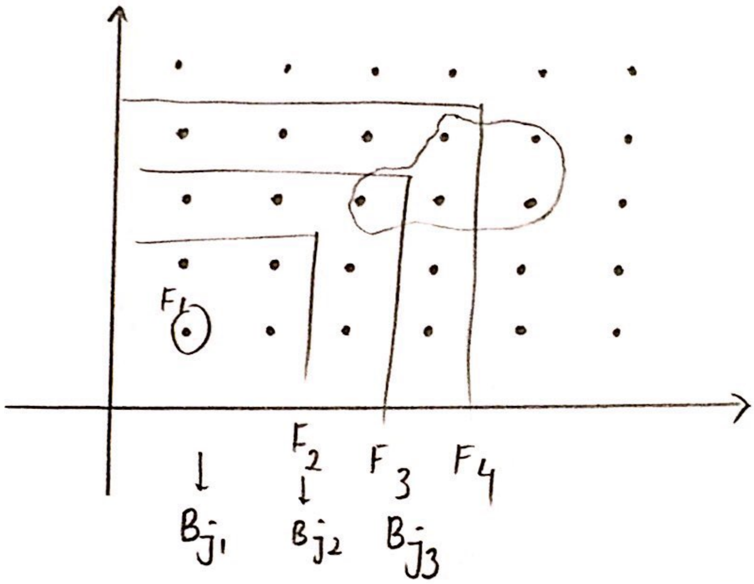


<u>Theorem</u> Let $\mathbb{N}^m = B_1 \cup B_2 \cup \cdots \cup B_q$ be a partition of $\mathbb{N}^m$. One of the sets B_j has the following property.

$$\forall F \subseteq \mathbb{N}^m \text{ finite}, \ \exists a \in \mathbb{N}, \exists \vec{b} \in \mathbb{N}^m : \ aF + \vec{b} \subseteq B_j.$$

<u>proof.</u>

First we will show that for a given $F \subseteq \mathbb{N}^m$ finite, there are B_j, $a \in \mathbb{N}$, $b \in \mathbb{N}^m$: $aF + \vec{b} \subseteq B_j$.



F_n is finite $\forall n$.

$$\bigcup_n F_n = \mathbb{N}^m$$

$F_n \subseteq F_{n+1}$ $\forall n \in \mathbb{N}$.

Let $\Lambda = \{1, 2, \cdots, q\}$, $\Omega = \Lambda^{\mathbb{N}^m}$.

$$d(\omega, \omega') = \inf \left\{ \frac{1}{k} : \omega(i_1, \dots, i_m) = \omega'(i_1, \dots, i_m) \text{ for any } 1 \leq i_1, \dots, i_m < k \right\}$$

<u>MBR:</u> X compact, $T_1, \dots, T_l$ homeomorphisms s.t. $T_i T_j = T_j T_i \Rightarrow \exists x \ \exists n_k \nearrow \infty$ $T_1^{n_k}(x) \to x$, $T_2^{n_k}(x) \to x$, $\dots$, $T_l^{n_k}(x) \to x$. $F \subseteq \mathbb{N}^m$ finite: $F = \{e_1, \dots, e_l\}$.

$T_i(\omega)(\vec{n}) = \omega(\vec{n} + e_i)$ $\forall i \leq l$.