21.09.2018
Arithmetic Seminar
Talk - 2

Van der Waerden's Theorem (1927)

Let $k, r \in \mathbb{N}$ be given. Then $\exists N = N(k,r) \in \mathbb{N}$:

when the integers in $[1, N]$ are colored using r colors, then there is a nontrivial monochoromatic k-term arithmetic progression (AP). That is,

$\exists\, a, b \neq 0$: $a, a+b, a+2b, \dots, a+(k-1)b$ all have the same color.

Theorem$'$: Let $r \in \mathbb{N}$ be given. If $\mathbb{N}$ is colored by using r colors, then for every $k \in \mathbb{N}$ there is a monochoromatic k-term AP.

Exercise: Deduce Theorem$'$ using the V.D.W. Theorem.

Erdős - Turan Conjecture (1935):

Let $k \in \mathbb{N}$, $\delta \in (0,1]$. Then $\exists N = N(k, \delta)$ such that for any $A \subseteq [1, N]$ with $|A| \geq \delta \cdot N$ A has k-term A.P.

(In other words, the probability of belonging to A of a positive integer from 1 to N is greater than δ, then A has a k-term AP.)

Exercise: Deduce Erdős - Turan Conjecture from Theorem'.

Roth's Theorem (1953)

There is a constant C such that $A \subseteq [1,N]$, $|A| \geq C \cdot \frac{N}{\log\log N} \Rightarrow A$ has a nontrivial 3-term AP.

Green - Tao Theorem (2003)

$\mathcal{P}$ contains arbitrary long AP's.

Idea for the proof of V.D.W.

Let $\Lambda = \{1, 2, \dots, r\}$ be the set of colors.
By a coloring of $\mathbb{Z}$ we mean a function $f: \mathbb{Z} \longrightarrow \Lambda$.
Let $X = \Lambda^{\mathbb{Z}}$ (the space of all coloring of $\mathbb{Z}$ with colors in Λ)

$f \in X \Rightarrow a, a+b, a+2b, \dots$ have the same color if $f(a) = f(a+b) = f(a+2b) = \dots f(a+(k-1)b)$.

$T: X \longrightarrow X$ is defined by $Tx(n) = x(n+1)$

$$f(a) = T^{b} f(a) = T^{2b} f(a) = \dots = T^{(k-1)b} f(a)$$

Poincare Recurrence:

Let X be a probability space with measure μ.
Let $T: X \to X$ be a measure-preserving map
$(\mu(T^{-1}(A)) = \mu(A) \quad \forall$ measurable $A)$

If $\mu(V) > 0$, $\exists x \in V$, $\exists n > 0$: $T^n(x) \in V$.

$V, T^{-1}(V), T^{-2}(V), T^{-3}(V), \dots$ can not be all disjoint. $\exists p \in T^{-n}(V) \cap T^{-m-n}(V) \Rightarrow T^n(p) \in V$, $T^{m+n}(p) \in V$.

$x := T^n(p) \Rightarrow x \in V$. $T^m(x) \in V$; $n.\delta > 1 \Rightarrow m \leq \frac{1}{\delta}$

An application:

$X = \mathbb{R}/\mathbb{Z}$, $V = \left[\frac{-1}{2\Theta}, \frac{1}{2\Theta}\right]$, $\theta \in (0, \infty)$ is fixed. Define $T: X \to X$, $T(x) = x + \theta$.

Dirichlet's theorem:

$\forall \theta \in \mathbb{R}$, $\forall \Theta \geq 1$ $\exists q \in \{1, 2, \dots, \Theta\}$: $\|q\theta\| \leq \frac{1}{\Theta}$

($\|q\theta\|$: the distance between $q\theta$ and the nearest integer).

Exercise: Show Dirichlet's theorem by using Poincare Recurrence.

Birkoff's Recurrence Theorem

Let X be a compact metric space, $T: X \longrightarrow X$ be a continuous map. There is a point $x \in X$, there is a sequence $n_k \longrightarrow \infty$ with $T^{n_k}(x) \longrightarrow x$.

Back to V. D. W. Theorem.

$$\Lambda = \{1, 2, \dots, r\}, \quad X = \Lambda^{\mathbb{Z}}$$

$$x, y \in X, \ x \neq y, \ d(x,y) = 2^{-m(x,y)}, \text{ where}$$

$$m(x,y) = \min\{\ell \geq 0 : x(\ell) \neq y(\ell) \text{ or } x(-\ell) \neq y(-\ell)\}$$

$$d(x,x) = 0.$$

Exercise: (X, d) is a compact metric space

$$T: X \longrightarrow X, \quad Tx(n) = x(n+1)$$

Exercise: T is continuous.

$x \in X$, $A := \overline{\{T^n(x)\}}$ $\Rightarrow$ A is $T^{\mathbb{N}}$-invariant, compact subspace $\Rightarrow \exists p \in A : p(0) = p(n) = p(2n) = \cdots = p(kn)$

$\Rightarrow \exists m \in \mathbb{Z} : T^m(x)$ and p agrees on $[-kn, kn]$

$\Rightarrow x(m) = x(m+2n) = \cdots = x(m+kn)$, $(k+1)$-term AP.

Deduction of the Finitary VDW Theorem:

Suppose not $\forall N$, $\exists$ an r-coloring f_N of $[-N, N]$ s.t. there is no monochoromatic k-term arithmetic progression. Extend each of these coloring to $\mathbb{Z}$. By compactness we may find a limit point of these elements. That limit point will define an element of X (a coloring of $\mathbb{Z}$ with r colors) (with no k-term AP).