The Class Number Formula and The Dedekind Zeta Function

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Overview

1. Number Fields
   - Ideals in Number Fields
   - Geometry of Number Fields
   - Dirichlet Unit Theorem
2. Analytic Class Number Formula
   - Explicit Class Number Formula

Number Fields

Definition

Any finite field extension K of $\mathbb{Q}$ is called a number field. So K is a field which is a $\mathbb{Q}-$vector space of finite dimension $[K:\mathbb{Q}]$.

Example

$\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}$. A number field of degree 2, an example of a quadratic number field.

K will denote a number field of degree $n = [K:\mathbb{Q}]$ for the rest of the presentation.

Definition

An element $\alpha \in \mathbb{C}$ is called an algebraic number if $f(\alpha) = 0$ for some $0 \neq f(X) \in \mathbb{Q}[X]$.

Note that every element of K is an algebraic number since K is a finite extension.

Definition

An element $\alpha \in \mathbb{C}$ is called an algebraic integer if $f(\alpha) = 0$ for some monic $0 \neq f(X) \in \mathbb{Z}[X]$.

Definition

Given an algebraic number α, the monic non-zero polynomial $f(X) \in \mathbb{Q}[X]$ with smallest degree so that $f(\alpha) = 0$ is called the minimal polynomial of α.

Proposition

The minimal polynomial of α has integer coefficients if and only if α is an algebraic integer.

Remark

The set of algebraic integers in $\mathbb{Q}$ is $\mathbb{Z}$.

Definition

The set of elements $\{\alpha \in K | \alpha \text{ is an algebraic integer } \}$ is called the ring of integers of K. It is denoted by $\mathcal{O}_K$.

Proposition

Let $\alpha \in K$. The following statements are equivalent:

1. The element α is an algebraic integer.
2. The abelian group $\mathbb{Z}[\alpha]$ is finitely generated.

Corollary

$\mathcal{O}_K$ is a ring:

Proof.

Since for any $\alpha, \beta \in \mathcal{O}_K$, $\mathbb{Z}[\alpha]$ and $\mathbb{Z}[\beta]$ are finitely generated. Then, so is $\mathbb{Z}[\alpha, \beta]$. $\mathbb{Z}[\alpha \pm \beta], \mathbb{Z}[\alpha\beta]$ are subgroups of $\mathbb{Z}[\alpha, \beta]$, therefore they are finitely generated and we are done.

Proposition

We have $\mathbb{Q}\mathcal{O}_K = K$.

Thus, for any $\alpha \in K$ we can find some $0 \neq z \in \mathbb{Z}$ such that $z\alpha \in \mathcal{O}_K$. Note that by the proposition, we can say that $Frac(\mathcal{O}_K) = K$.

Proposition

Given a number field K of degree n over $\mathbb{Q}$, it has $n-$many distinct embeddings into $\mathbb{C}$.

Let K be a number field of degree n and let $\sigma_1, \ldots, \sigma_n$ be the distinct monomorphisms of K into $\mathbb{C}$. Take any $\alpha \in K$. Then,

$$N_{K/\mathbb{Q}}(\alpha) = \prod_{i=1}^{n} \sigma_i(\alpha)$$

.

Remark

If α is an algebraic integer, then its norm belongs to $\mathbb{Z}$ and α is a unit if and only if $N_{K/\mathbb{Q}}(\alpha) = \pm 1$.

Proposition

Given a number field K, $\mathcal{O}_K$ is a free abelian group of rank n and it is integrally closed.

Since $\mathcal{O}_K$ is a free abelian group of rank n, it has a $\mathbb{Z}$-basis, say $\alpha_1, \ldots, \alpha_n$. Define an $n \times n$ matrix M as:

$$M := (\sigma_i(\alpha_j))_{1 \le i,j \le n}.$$

Definition

Discriminant Δ_K of a number field is defined as $\det(M^2)$.

Remark

The discriminant is independent from the choice of basis.

Ideals in Number Fields

Let K be a number field of finite degree n and let $I \subseteq \mathcal{O}_K$ be a non-zero ideal. The norm of I, $N(I)$ is defined as $|\mathcal{O}_K/I|$. It is also multiplicative.

Proposition

Let I be a non-zero ideal of $\mathcal{O}_K$. Then the norm of I is finite and for any $\alpha \in \mathcal{O}_K$, $N_{K/\mathbb{Q}}(\alpha\mathcal{O}_K) = |N_{K/\mathbb{Q}}(\alpha)|$.

Note that if we take an ascending chain of ideals $(I_i)_{i\in\mathbb{N}}$, then it has to be stationary since $N(I_1) > N(I_2) > \cdots > 0$. Therefore, we can say that $\mathcal{O}_K$ is a Noetherian ring.

Proposition

If $0 \neq \mathfrak{p}$ is a prime ideal of $\mathcal{O}_K$, then it is maximal. Therefore, $\mathcal{O}_K/\mathfrak{p}$ is a field and $|\mathcal{O}_K/\mathfrak{p}| = N(\mathfrak{p})$ is finite. Thus, we can say that $N(\mathfrak{p}) = p^k$ for some prime number p and $k \in \mathbb{Z}^{>0}$.

Remark

We know that $\mathcal{O}_K$ is a integrally closed, Noetherian domain such that its non-zero prime ideals and maximal ideals coincide. Therefore, $\mathcal{O}_K$ is a *Dedekind domain*.

Definition

Let us call the ideals of $\mathcal{O}_K$ as *integral ideals*. Suppose that I is a $\mathcal{O}_K$ *– submodule* of K so that for some $0 \neq \alpha \in \mathcal{O}_K$, $\alpha I \subseteq \mathcal{O}_K$. Then, I is called a *fractional ideal* of $\mathcal{O}_K$. Any integral ideal is a fractional ideal. Ideal operations on fractional ideals is defined similarly. Fractional ideals gives a monoid under multiplication with identity element $\mathcal{O}_K$(In fact, a group).

For any prime ideal $0 \neq \mathfrak{p}$, define $\mathfrak{p}^{-1} = \{x \in K | x\mathfrak{p} \subseteq \mathcal{O}_K\}$. Then,

1. $\mathfrak{p}^{-1}$ is a fractional ideal.
2. $\mathcal{O}_K \subsetneq \mathfrak{p}^{-1}$.
3. $\mathfrak{p}^{-1}\mathfrak{p} = \mathcal{O}_K$.

Proposition

Every non-zero integral ideal $I \subseteq \mathcal{O}_K$ can be written uniquely as $\mathfrak{p}_1 \dots \mathfrak{p}_k$ for some prime ideals $\mathfrak{p}_1 \dots \mathfrak{p}_k$.
Furthermore, for a non-zero fractional ideal I, it can be written as $\mathfrak{p}_1 \dots \mathfrak{p}_k \mathfrak{q}_1^{-1} \dots \mathfrak{q}_l^{-1}$ for some prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_k, \mathfrak{q}_1, \dots, \mathfrak{q}_l$.

Definition

Let us denote the group of fractional ideals by I_K. As we define principal ideals, we define principal fractional ideals. I is a *principal fractional ideal* if it is of the form $x\mathcal{O}_K$ for some $x \in K^\times$. The set of principal fractional ideals are denoted as P_K and P_K is a subgroup of I_K. Then, define the quotient group $Cl(K) = I_K/P_K$. It is called the *ideal class group of K*. The equivalence relation $\sim$ on $Cl(K)$ is given as $I \sim J$ if $\alpha I = J$ for some $\alpha \in K$.

Class number

The cardinality of $Cl(K)$ is called the class number of K and denoted by h_K.

Proposition

$h_K = 1 \iff \mathcal{O}_K$ is UFD.

Proof.

($\Longleftarrow$) Since any integral ideal has a prime ideal decomposition, if we show that prime ideals are principal, we are done. Now, suppose that $\mathcal{O}_K$ is a UFD and let $\mathfrak{p}$ be any non-zero prime ideal of $\mathcal{O}_K$. Now take any non-zero element $p \in \mathfrak{p}$. Since $\mathcal{O}_K$ is UFD, we can write $p = \pi_1^{e_1} \dots \pi_m^{e_m}$ for some irreducible elements π_i. Then, $\pi_i \in \mathfrak{p}$ for some $i \in \{1, \dots, m\}$. In a UFD, irreducible elements are prime and $(\pi_i) \subseteq \mathfrak{p}$ is a prime ideal. Also, since prime ideals are maximal, we have $\mathfrak{p} = (\pi_i)$.

□

Remark

Recall that we have n embeddings into $\mathbb{C}$. We say that σ is a complex embedding if $\sigma(K) \subsetneq \mathbb{R}$. Otherwise, we call it as a real embedding. Complex embeddings come in pairs $\sigma, \overline{\sigma}$, therefore we have r_1 real and $2r_2$ complex embeddings such that $r_1 + 2r_2 = n$.

$\mathcal{O}_K$ can be embedded inside $\mathbb{R}^n$ and its image gives a lattice. Geometric arguments give us the following results:

Theorem

Every ideal class of K contains an integral ideal I such that

$$N(I) \leq \left(\frac{4}{\pi}\right)^{r_2} \frac{n!}{n^n} |\Delta_K|^{1/2}.$$

This bound is called *Minkowski's bound* in the literature.

Lemma

There are finitely many integral ideals with norm bounded by b for any $b \in \mathbb{N}$.

Proof.

Let $I \subseteq \mathcal{O}_K$ be a non-zero integral ideal with $N(I) < b$. Then we have $I = \mathfrak{p}_1^{e_1} \dots \mathfrak{p}_m^{e_m}$ for some prime ideals $\mathfrak{p}_i$ and $e_i \in \mathbb{Z}^{>0}$. Note that each prime factor $\mathfrak{p}_i$ comes from a prime number p such that $\mathfrak{p}_i | p\mathcal{O}_K$. Also, we have $N(I) = N(\mathfrak{p}_1)^{e_1} \dots N(\mathfrak{p}_m)^{e_m}$ where $\mathfrak{p}_i$ has norm p^k for some prime number p and $k \in \mathbb{Z}^{>0}$ such that k is bounded by n. Since the possible prime numbers p are bounded by b and the possible prime ideals that can appear in a factorization are also bounded, there can be only finitely many ideals with bounded norm. □

Corollary

Class number h_K of K is finite.

Proof

We know that an integral ideal can be chosen from each ideal class of K so that its norm is bounded by Minkowski's bound. We know also that the ideals with a bounded norm is a finite set. Thus, h_K is finite.

Dirichlet Unit Theorem

Theorem (Dirichlet Unit Theorem)

$$\mathcal{O}_K^\times \cong \mathbb{Z}^r \times \mu(\mathcal{O}_K)$$

where $r = r_1 + r_2 - 1$ *and* $\mu(\mathcal{O}_K)$ *is the group of roots of unity in* $\mathcal{O}_K$.

Remark

$\mu(\mathcal{O}_K)$ is a finite group. So, by Dirichlet unit theorem we have $\epsilon_1, \ldots, \epsilon_r \in \mathcal{O}_K^\times$ such that for any $\epsilon \in \mathcal{O}_K^\times$, $\epsilon = \zeta\epsilon_1^{e_1} \ldots \epsilon_r^{e_r}$ for some $\zeta \in \mu(K)$ and $e_1, \ldots, e_n \in \mathbb{Z}$. $\epsilon_1, \ldots, \epsilon_r$ are called fundamental units.

Now, we give the analytic class number formula which consists of important invariants of a ring of integers. Let us recall the Riemann zeta function first.

Definition (The Riemann zeta function)

The Riemann zeta function $\zeta(s)$ is defined as

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

for any $s \in \mathbb{C}$ with $Re(s) > 1$.

Now, let us define the Dedekind zeta function for a number field, which generalizes the Riemann zeta function.

Definition

Let K be a number field. Then, the Dedekind zeta function of K is defined as

$$\zeta_K(s) = \sum_{0 \neq I \subseteq \mathcal{O}_K} \frac{1}{N(I)^s}.$$

Remark

If we take $K = \mathbb{Q}$, we have $N_{K/\mathbb{Q}}(I)^s = n^s$ for some $n \in \mathbb{Z}^{>0}$ since any ideal $0 \neq I \subseteq \mathbb{Z}$ is of the form $n\mathbb{Z}$. Thus, $\zeta_{\mathbb{Q}}(s) = \zeta(s)$.

Since we can factorize the ideals of $\mathcal{O}_K$ uniquely, the Dedekind zeta function has the following Euler product when $Re(s) > 1$:

$$\zeta_K(s) = \prod_{0 \neq \mathfrak{p} \subseteq \mathcal{O}_K, prime} \frac{1}{1 - N(\mathfrak{p})^{-s}}.$$

Regulator of K

Recall that we have $r_1 + 2_{r_2}$ many embeddings of K into $\mathbb{C}$. Let us say that we have the real embeddings $\rho_1, \ldots, \rho_{r_1}$ and non-conjugate complex embeddings $\sigma_1, \ldots, \sigma_{r_2}$. Define

$$\lambda : K^\times \to \mathbb{R}^{r_1+r_2}$$

$$x \mapsto (\log(|\rho_1(x)|), \ldots, \log(|\rho_{r_1}(x)|), \log(|\sigma_1(x)|^2), \ldots, \log(|\sigma_{r_2}(x)|^2)).$$

For simplicity, say that $\lambda(x) = (\lambda_1(x), \ldots, \lambda_{r_1+r_2}(x))$. Define a matrix $A_{ij} = (\lambda_i(\epsilon_j))$ where $1 \leq i \leq r_1 + r_2$ and $1 \leq j \leq r_1 + r_2 - 1$ so that ϵ_j's are the fundamental units. Take any $r_1 + r_2 - 1 \times r_1 + r_2 - 1$-minor of this matrix and take determinant. Regulator of K, R_K is defined as the absolute value of the resulting determinant.

R_K is well defined, it is independent of the choice of the row that we delete.

Example

Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic number field for some square-free $d \in \mathbb{Z}$.
Then, by Dirichlet's Unit theorem we deduce the following:
If K is real quadratic, then we have one fundamental unit and if K is imaginary quadratic then there is not any fundamental unit.
Let K be a real quadratic field with fundamental unit ϵ. Then, the corresponding matrix is

$$\begin{bmatrix} \rho_1(\epsilon) \\ \rho_2(\epsilon) \end{bmatrix}.$$

In the end, deleting the first or the second row gives the same value since

$$\Big|\log|\rho_1(\epsilon)|\Big| = \Big|\log|\frac{1}{\rho_2(\epsilon)}|\Big| = \Big|-\log|\rho_2(\epsilon)|\Big| = \Big|\log|\rho_2(\epsilon)|\Big|.$$

Thus, $R_K = \log(\epsilon)$, provided that $\epsilon > 0$.
Regulator of an imaginary quadratic field is 1 since the corresponding matrix is 0 by 0 which has determinant 1.

Finally, we are ready to give the Analytic Class Number Formula:

Theorem (Analytic Class Number Formula)

The Dedekind zeta function $\zeta_K(s)$ converges for any s with $Re(s) > 1$. It has a simple pole at $s = 1$ and

$$\lim_{s \to 1} (s-1)\zeta_K(s) = \frac{2^{r_1+r_2}\pi^{r_2} R_K}{|\mu(K)|\sqrt{|\Delta_K|}} \; h_K.$$

Now, let us see an example. Before that, we need the following theorem:

Theorem

Let $X \subseteq \mathbb{R}^n$ be a cone. Assume that $f : X \to \mathbb{R}^{>0}$ is a function satisfying $f(cx) = c^n f(x)$ for any $x \in X$ and $c \in \mathbb{R}^{>0}$.
Let the set $U = \{x \in X | f(x) \leq 1\}$ be bounded with volume $\omega = vol(U)$ such that $\omega \neq 0$ and let Γ be a lattice in $\mathbb{R}^n$ with volume $\upsilon = vol(\Gamma)$.

Then, $z(s) = \sum_{\Gamma \cap X} \frac{1}{f(x)^s}$ converges for $Re(s) > 1$ and we have

$$\lim_{s \to 1} (s-1) z(s) = \frac{\omega}{\upsilon}.$$

.

Fact

A cone $X \subseteq \mathbb{R}^n$ is a subset satisfying $cx \in X$ for any $c \in \mathbb{R}$ and $x \in X$. A lattice Γ inside $\mathbb{R}^n$ is a subgroup of $\mathbb{R}^n$ which is discrete and has a $\mathbb{Z}$-basis.

Now, let us find the residue of the Dedekind zeta function of $K = \mathbb{Q}(i)$.

First, note that since K is an imaginary quadratic field we have $r_1 = 0, r_2 = 1$ and $\mu(K) = \{\pm 1, \pm i\}$. Also, we have $R_K = 1$ and $\Delta_K = -4$. Finally, we have $h_K = 1$. Then,

$$\frac{2^{r_1+r_2}\pi^{r_2}R_K}{|\mu(K)|\sqrt{|\Delta_K|}} = \frac{2\pi}{4\sqrt{|-4|}} = \frac{\pi}{4}.$$

Therefore, we expect to see that the residue of ζ_K at $s = 1$ is $\dfrac{\pi}{4}$.

To use the theorem, let us define the cone $X \subseteq \mathbb{R}^2$ to be $\mathbb{R}^2 - (0,0)$. Also, define $f : X \to \mathbb{R}^{>0}$ to be $f((a,b)) = a^2 + b^2$. Note that for any $c \in \mathbb{R}^{>0}$ and $x \in X$,we have $f(cx) = c^2 x$. Then, define U to be the set

$$\{x \in X | f(x) \leq 1\} = \{(a,b) \in \mathbb{R}^2 | a^2 + b^2 \leq 1\}.$$

U a unit square in $\mathbb{R}^{>0}$ therefore its volume is $\pi \neq 0$. Lastly, define Γ to be $\mathbb{Z}^2$ which is a lattice in $\mathbb{R}^2$ with volume 1.

Then, $z(s) = \sum_{\Gamma \cap X} \frac{1}{f(x)^s} = \sum_{(a,b) \in \mathbb{Z}^2 - (0,0)} \frac{1}{(a^2 + b^2)^s}$.

In this example, $\mathcal{O}_K = \mathbb{Z}[i]$ and since $h_K = 1$, every ideal is principal and of the form $(a + bi)\mathcal{O}_K$ for some $a + bi \in \mathbb{Z}[i]$. Note that the norm of the ideal $(a + bi)\mathcal{O}_K$ is $N((a + bi)\mathcal{O}_K) = |N_{K/\mathbb{Q}}(a + bi)| = a^2 + b^2$. Furthermore, recall that we have 4 units $\{\pm 1, \pm i\}$ in $\mathcal{O}_K$ so that $(a + bi), -(a + bi), i(a + bi), -i(a + bi)$ generate the same ideal in $\mathcal{O}_K$. Thus, to represent an ideal $I \subseteq \mathcal{O}_K$ as $< a + bi >$, we can consider only the pairs (a, b) inside the same quadrant in $\mathbb{Z}^2$.

Therefore,

$$\zeta_{\mathbb{Q}(i)}(s) = \sum_{0 \neq I \subseteq \mathbb{Z}[i]} \frac{1}{N_{K/\mathbb{Q}}(I)^s} = \sum_{(a,b) \in \mathbb{Z}^{>0} \times \mathbb{Z}^{>0}} \frac{1}{(a^2+b^2)^s}$$

$$= \frac{1}{4} \sum_{(a,b) \in \mathbb{Z}^2 - (0,0)} \frac{1}{(a^2+b^2)^s} = \frac{1}{4} z(s).$$

Then, by the theorem we know that $z(s)$ converges for $Re(s) > 1$ and we have

$$\lim_{s \to 1} (s-1) z(s) = \frac{vol(U)}{vol(\Gamma)}) = \frac{\pi}{1}.$$

Thus, we conclude that $\zeta_{\mathbb{Q}(i)}(s) = \frac{1}{4} z(s)$ converges for $Re(s) > 1$ and

$\lim_{s \to 1} (s-1) \zeta_{\mathbb{Q}(i)}(s) = \frac{\pi}{4}$ as we expected by the *Class Number Formula*.

Let us compute the class number of a quadratic number field.

Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic number field for some d, a square-free integer. Then, we have the following:

1. $\mathcal{O}_K = \mathbb{Z}[\sqrt{d}]$ and $\Delta_K = 4d$, if $d \equiv 2, 3 \pmod 4$.
2. $\mathcal{O}_K = \mathbb{Z}[\frac{1+\sqrt{d}}{2}]$ and $\Delta_K = d$ if $d \equiv 1 \pmod 4$.

Now take any prime number p. $p\mathcal{O}_K$ is an ideal of $\mathcal{O}_K$ generated by p, so it can be written as a product of prime ideals. We have 3 cases:

1. p *splits* in K if $p\mathcal{O}_K = P'Q'$ for some distinct prime ideals P', Q' of norm p.
2. p is inert in K if $p\mathcal{O}_K$ is a prime ideal of $\mathcal{O}_K$ of norm p^2.
3. p is ramifed in K if $p\mathcal{O}_K = P^2$ for some prime ideal P of $\mathcal{O}_K$ of norm p.

A *Dirichlet character modulo m* is a group homomorphism

$$\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C} - 0.$$

It can be extended to $\mathbb{Z}/m\mathbb{Z}$ by setting $\chi(a) = 0$ for any $a \in \mathbb{Z}/m\mathbb{Z} - (\mathbb{Z}/m\mathbb{Z})^\times$. We call χ as principal if it is the trivial homomorphism and non-principal otherwise.

We can extend the domain of χ to whole positive integers. If $n \in \mathbb{Z}^{>0}$ then define $\chi(n)$ to be $\chi(\bar{n})$ where $\bar{n}$ is the congruence class of n in $(\mathbb{Z}/m\mathbb{Z})$.

We define the *Dirichlet L-function* with Dirichlet character χ as

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

for $Re(s) > 1$.
Now, suppose that K is a quadratic field. Recall that a prime number either splits, ramifies or is inert. Define

$$\chi(p) = \begin{cases} 1 & \text{if p splits} \\ -1 & \text{if p is inert} \\ 0 & \text{if p ramifies} \end{cases}$$

Then, χ gives a real character modulo $|\Delta_K|$. Actually, this character is the Kronecker symbol $(|\Delta_K|, .)$

For this character, we have

$$\zeta_K(s) = \zeta(s)L(s, \chi).$$

Then, if we multiply both sides with $(s-1)$ and take limit as s goes to 1:

$$\frac{2^{r_1+r_2}\pi^{r_2}R_K}{|\mu(K)|\sqrt{|\Delta_K|}}\; h_K = L(1, \chi).$$

Now, if K is a real quadratic field, then we have $r_1 = 2, r_2 = 0$, $|\mu(K)| = 2$, and $R_K = \eta$ where η is a fundamental unit. Thus, we have:

$$h_K = \frac{\sqrt{|\Delta_K|}}{2\log \eta}\ L(1,\chi).$$

On the other hand, if K is an imaginary quadratic field, then we have $r_1 = 0, r_2 = 1$ and $R_K = 1$. Therefore, we conclude:

$$h_K = \frac{|\mu(K)|\sqrt{|\Delta_K|}}{2\pi}\ L(1,\chi).$$

Therefore, the value of $L(1,\chi)$ can be useful to make assumption on h_K. However, note that it is not easy to compute $L(1,\chi)$, sometimes. Lastly, in our case $L(1,\chi) \neq 0$. This plays a fundamental role on proving Dirichlet's theorem on arithmetic progressions. Thus, we can make some observations on the density of primes via *the class number formula*.

Theorem (Dirichlet's Theorem on Arithmetic Progressions)

Let $a, b \in \mathbb{Z}^{>0}$ such that $(a, b) = 1$. Then, there exist infinitely many prime numbers p of the form $an + b$ for some $n \in \mathbb{Z}^{>0}$.

Now, let us show that the class number of $\mathbb{Q}(\sqrt{2})$ with our explicit class number formula.

Example

Let $K = \mathbb{Q}(\sqrt{2})$ with fundamental unit $\eta = 1 + \sqrt{2}$. We have $\Delta_K = 8$. Thus,

$$h_K = \frac{\sqrt{8}}{2\log(1+\sqrt{2})} \; L(1,\chi) \approx 1.605 \;\; L(1,\chi).$$

and we have

$$L(1,\chi) = 1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} - \dots$$

However, the value of $L(1,\chi)$ is between $1 - \frac{1}{3}$ and $1 - \frac{1}{3} + \frac{1}{5}$ so that $L(1,\chi) < 1$.
Thus, $h_K < 1.605$. Therefore, h_K must be 1.

Final Notes

In this thesis, we deal with the algebraic side of the *Dedekind Zeta Function*. It has also an analytic side. The Dedekind zeta function can be extended by analytic continuation to whole complex plane.

Conjecture (Riemann Hypothesis)

The real part of any non-trivial zero of $\zeta(s)$ is $\frac{1}{2}$.

Clay Math Institute will pay 1 million dollars to the one who proves or disproves the conjecture(8^{th} Problem of Hilbert).

In terms of L-functions, the hypothesis can be formulated as:

Conjecture (Generalized Riemann Hypothesis)

The generalized Riemann hypothesis states that for every Dirichlet character χ and every $s \in \mathbb{C}$ with $L(s, \chi) = 0$, if $0 < Re(s) < 1$, then $Re(s) = \frac{1}{2}$.

Lastly, the conjecture can be formulated for Dedekind zeta function as below:

Conjecture (Extended Riemann Hypothesis)

For every number field K and $s \in \mathbb{C}$ with $\zeta_K(s) = 0$, if the $0 < Re(s) < 1$, then $Re(s) = \frac{1}{2}$.

References

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$$\zeta(\text{Teşekkürler}) = 0$$

$$\implies \mathit{Re}(\text{Teşekkürler}) = \frac{1}{2}$$