DOKUZ EYLUL UNIVERSITY FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS

STUDENT SEMINARS

Cardano's Formula and Casus Irreducibilis

Çağdaş Çiğdemoğlu

Dokuz Eylül University

ABSTRACT

We will start with a **general monic cubic equation** in the form

$$x^3 + bx^2 + cx + d = 0,$$

and transform it into the following form

$$y^3 + py + q = 0,$$

using a substitution. Then, we will construct the **Cardano Formulas** to find the roots of the equation. The roots are expressed as follows:

$$y_1 = \sqrt[3]{\frac{-q + \sqrt{q^2 + \frac{4p^3}{27}}}{2}} + \sqrt[3]{\frac{-q - \sqrt{q^2 + \frac{4p^3}{27}}}{2}},$$

$$y_2 = \omega\sqrt[3]{\frac{-q + \sqrt{q^2 + \frac{4p^3}{27}}}{2}} + \omega^2\sqrt[3]{\frac{-q - \sqrt{q^2 + \frac{4p^3}{27}}}{2}},$$

$$y_3 = \omega^2\sqrt[3]{\frac{-q + \sqrt{q^2 + \frac{4p^3}{27}}}{2}} + \omega\sqrt[3]{\frac{-q - \sqrt{q^2 + \frac{4p^3}{27}}}{2}},$$

where $\omega = e^{i\frac{2\pi}{3}} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ is a **primitive cube root of unity** and where the above cube roots are one of the three complex cube roots whose product is $-\frac{p}{3}$ and these are fixed in the above formulas.

We will learn about the **discriminant** Δ of the above monic cubic polynomial, understand its significance, and examine how the roots change depending on the value of the discriminant.

For the monic cubic polynomial in the form $y^3 + py + q$, the discriminant is expressed by:

$$\Delta = -27q^2 - 4p^3 = (y_1 - y_2)^2(y_1 - y_3)^2(y_2 - y_3)^2.$$

For the general monic cubic polynomial $x^3 + bx^2 + cx + d$, whose roots are x_1, x_2, x_3, the discriminant is expressed by:

$$\Delta = b^2c^2 + 18bcd - 4c^3 - 4b^3d - 27d^2 = (x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2.$$

We will also discuss *Casus Irreducibilis*, which occurs when the coefficients b, c, d (or p, q) are in a subfield of F of $\mathbb{R}$, the cubic polynomial is irreducible over F (equivalently, the cubic equation has no roots in F) and when the discriminant is positive. In this case, the cubic equation has three distinct real roots that cannot be expressed using **real radicals**.

This seminar serves as an introduction to my project, which focuses on understanding the proof of *Casus Irreducibilis* using **Galois Theory**.

DATE & TIME: **18 Dec 2024, Wednesday at 15:00**

CLASSROOM: **B255**