DOKUZ EYLUL UNIVERSITY FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS

STUDENT SEMINARS

Fundamental Theorem of Symmetric Polynomials, Newton's Identities and Discriminants

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ABSTRACT

We will define **symmetric polynomials** and the **elementary symmetric polynomials** in n indeterminates over a field F. The elementary symmetric polynomials in the indeterminates $x_1, x_2, \ldots, x_n$ are as follows:

$$\begin{aligned} \sigma_1 &= x_1 + x_2 + \ldots + x_n \\ \sigma_2 &= \sum_{1 \le i < j \le n} x_i x_j \\ &\vdots \\ \sigma_n &= x_1 x_2 \ldots x_n \end{aligned}$$

The *Fundamental Theorem of Symmetric Polynomials* states that any symmetric polynomial can be expressed as a polynomial in the elementary symmetric polynomials, that is:

Theorem. *Let $f(x_1, x_2, \ldots, x_n)$ be a symmetric polynomial in the n indeterminates $x_1, x_2, \ldots, x_n$ over a field F. Then, there exists a polynomial $g(y_1, y_2, \ldots, y_n)$ in the n indeterminates $y_1, y_2, \ldots, y_n$ such that*

$$f(x_1, x_2, \ldots, x_n) = g(\sigma_1, \sigma_2, \ldots, \sigma_n),$$

where $\sigma_1, \sigma_2, \ldots, \sigma_n$ are the above elementary symmetric polynomials of the n indeterminates $x_1, x_2, \ldots, x_n$. Moreover, the polynomial $g(y_1, y_2, \ldots, y_n)$ is uniquely determined.

We will prove this theorem using the *graded lexicographic order* for multivariable polynomials.

Using the recurrence relation from the *Newton Identities*, we will learn how to express the sum of powers of the indeterminates, that is, the polyomials

$$s_k = x_1^k + x_2^k + \ldots + x_n^k$$

for a positive integer k, as polynomials in terms of the elementary symmetric polynomials. We will reinforce this understanding with examples.

The discriminant in the indeterminates $x_1, x_2, \ldots, x_n$ over the field F is given by:

$$\Delta = \prod_{1 \le i < j \le n} (x_i - x_j)^2 \in F[x_1, \ldots, x_n].$$

The discriminant is a symmetric polynomial, and we will express it in terms of the elementary symmetric polynomials using determinants.

This seminar, as part of my graduation project titled *Symmetric Polynomials, Newton's Identities, Discriminants, and Resultants*, serves as an introduction to a method for calculating the **discriminant** (Δ) of an n-th degree polynomial without finding its roots.

DATE & TIME: **26th December 2024, Thursday, at 15:00**

CLASSROOM: **B254**