Cohen–Macaulayness and Gorensteiness of the Cozero-Divisor Graph of $\mathbb{Z}_n$

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Outline

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- There are many vanishing and exactness results that are really useful, and they are valid when the ring is Cohen-Macaulay. They were main tools in the research of the people in commutative algebra in the last decade.
- Recently, the combinatorial methods have been employed for studying these rings.

Definition (Krull dimension)

The Krull dimension of a ring R is defined to be the supremum length of proper chains of prime ideals in R, that is, the maximum n such that there exists a sequence of prime ideals $p_0 \subset p_1 \subset \ldots \subset p_n$ in R.

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Example

- Krull dimension of polynomial ring in one variable over real numbers, $\mathbb{R}[x]$ is 1. ($\{0\} \subset \langle x \rangle$ is proper chain of prime ideals in $\mathbb{R}[x]$.)

Definition (R-regular sequence)

A sequence of elements of a Noetherian local ring R is called an R-regular sequence if it is a sequence of nonzero elements $x_1, x_2, \ldots, x_n$ in the maximal ideal of R such that x_1 is not a zero divisor of R and x_k is not a zero divisor of $\frac{R}{\langle x_1, x_2, \ldots x_{k-1} \rangle}$ for $k = 2, \ldots, n$.

Definition (Depth)

Depth of a Noetherian local ring R, denoted by depth R, is the maximum n such that there exists a sequence of nonzero elements $x_1, x_2, \ldots, x_n$ *in the maximal ideal of R such that* x_1 *is not a zero divisor of R and* x_k *is not a zero divisor of* $\frac{R}{\langle x_1, x_2, \ldots x_{k-1} \rangle}$ *for* $k = 2, \ldots, n$.

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Example

All fields and finite commutative rings with unity have depth zero. In $\mathbb{R}[x]; < x >$is a maximal ideal and x is not a zero divisor of $\mathbb{R}[x]$. It follows that the localization A of $\mathbb{R}[x_1, \ldots, x_n]$ at the maximal ideal $m = \langle x_1, \ldots, x_n\rangle$ has depth at least n. In fact, A has depth equal to n; that is, there is no regular sequence in the maximal ideal of length greater than n.

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Definition (Cohen–Macaulay ring)

Also, a Noetherian ring is said to be Cohen–Macaulay if all of its localizations at all maximal ideals are Cohen–Macaulay.

- Graphs and simplicial complexes are widely used structures in the characterization of Cohen–Macaulay rings, edge ideals/Stanley-Reisner ideals acts as strong connecting tool between graphs and monomial ideals.
- Richard P. Stanley is well known for his fundamental and important contributions to combinatorics and its relationship to algebra and geometry, in particular in the theory of simplicial complexes.
- To each simpicial complex Δ, Stanley associated a quotient ring $k[\Delta]$, called Stanley-Reisner ring (or face ring) of Δ in such a manner that the combinatorial properties of the simpicial complex Δ are intimately related with the algebraic properties of the Stanley-Reisner ring $k[\Delta]$.

Definition

A simplicial complex Δ *on a vertex set* $V = \{x_1, \ldots, x_n\}$ *is a set of subsets of* V *that satisfies the following:*

(i) *If* $F \in \Delta$ *and* $G \subseteq F$*, then* $G \in \Delta$*;*

(ii) *For each* $i = 1, \ldots, n$, $\{x_i\} \in \Delta$*. The elements of* Δ *are called its faces.*

Definition

Let $R = K[x_1, \ldots, x_n]$ *be a polynomial ring over a field* K*. The Stanley Reisner ideal* I_Δ *defined as*
$I_\Delta = (\{x_{i_1} \cdot x_{i_2} \cdots x_{i_r} \mid i_1 < i_2 \ldots < i_r, \{x_{i_1}, x_{i_2}, \ldots x_{i_r}\} \notin \Delta\})$,
and its Stanley-Reisner ring $k[\Delta]$ *is defined as the quotient ring* R/I_Δ.

Example

The Stanley-Reisner ideal of C_4 is $\langle x_0x_1, x_1x_2, x_2x_3, x_3x_0\rangle$ and stanley-Reisner ring is $\dfrac{K[x_0, x_1, x_2, x_3]}{\langle x_0x_1, x_1x_2, x_2x_3, x_3x_0\rangle}$, where K is any field.

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- *Let G be a simple graph whose vertex set $V(G) = \{x_1, x_2, \ldots, x_n\}$ and $K[x_1, x_2, \ldots, x_n]$ be a polynomial ring on n variables over a field K. The edge ideal of G, denoted by $I(G)$, is the ideal generated by all the edges of G in $K[x_1, x_2, \ldots, x_n]$. That is $I(G) = \langle x_i x_j | \{x_i, x_j\} \in E(G) \rangle \subset K[x_1, x_2, \ldots, x_n]$.*

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- *The quotient ring $K[x_1, x_2, \ldots, x_n]/I(G)$ is called the edge ring of the graph G.*

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- A graph G is Cohen-Macaulay then it is well covered.

Definition (Vertex decomposable)

A simplicial complex Δ *is vertex decomposable if either* Δ *is a simplex, or* $\Delta = \emptyset$*, or* Δ *contains a vertex* v*, called a shedding vertex, such that both the link,*$\mathrm{lk}_{\Delta}(v)$ *and the deletion,* $\mathrm{del}_{\Delta}(v)$ *are vertex decomposable, and such that every facet of* $\mathrm{del}_{\Delta}(v)$ *is a facet of* Δ*. .*

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Definition (Link of a Face)

Let σ *be a face of the simplicial complex* Δ*. The link of* σ *is defined as:*

$$\mathrm{link}_{\Delta}(\sigma) = \{\tau \in \Delta \mid \tau \cup \sigma \in \Delta \textit{ and } \tau \cap \sigma = \emptyset\}.$$

Definition (Deletion of a Face)

If Δ is a simplicial complex and v is a vertex of Δ, the deletion of v, denoted by $\mathrm{del}_\Delta(v)$, is the subcomplex consisting of the faces of Δ that do not contain v.

Proposition (Vertex decomposable)

Let Δ *be a simplicial complex with* $\dim \Delta = 1$. *Then* Δ *is vertex-decomposable (Cohen–Macaulay) if and only if* Δ *is connected.*

Definition (Goreenstein Ring)

A Noetherian local ring $(R, \mathfrak{m})$ *is Gorenstein if*

$$\operatorname{injdim}_R R < \infty.$$

More generally, a Noetherian ring R *is Gorenstein if* $R_{\mathfrak{m}}$ *is Gorenstein for all* $\mathfrak{m} \in \operatorname{Max} R$.

Proposition (Gorenstein)

Let Δ be a simplicial complex. Then Δ is Gorenstein if and only if $\mathrm{core}(\Delta)$ *is an Eulerian complex which is Cohen–Macaulay.*

Let Δ be a simplicial complex of dimension $d-1$ and let f_i denote the number of faces of Δ of dimension i. The sequence $f(\Delta) = (f_0, f_1, \dots, f_{d-1})$ is called the *f-vector* of Δ. Letting $f_{-1} = 1$, the *reduced Euler characteristic* of Δ, denoted by $\widetilde{\chi}(\Delta)$, is defined to be

$$\widetilde{\chi}(\Delta) = \sum_{i=-1}^{d-1} (-1)^i f_i.$$

We call Δ *Eulerian* if it is pure and $\widetilde{\chi}(\mathrm{lk}_\Delta(F)) = (-1)^{\dim \mathrm{lk}_\Delta(F)}$ holds true for all $F \in \Delta$.

Definition (Cozero divisor Graph)

Let R be a ring with unity. The cozero-divisor graph of a ring R, denoted by $\Gamma'(R)$, is an undirected simple graph whose vertices are the set of all non-zero and non-unit elements of R, and two distinct vertices x and y are adjacent if and only if

$$x \notin Ry \quad \text{and} \quad y \notin Rx.$$

Structure of $\Gamma'(Z_n)$

- Let $d_1, d_2, \ldots, d_\ell$ be the proper divisors of n. For $1 \leq j \leq \ell$, consider the following sets:

$$A_{d_j} = \{x \in \{1, \ldots, n-1\} : \gcd(x, n) = d_j\}.$$

Then the sets $A_{d_1}, A_{d_2}, \ldots, A_{d_\ell}$ form a partition of the vertex set of the graph $\Gamma'(Z_n)$.

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- For $x, y \in A_{d_j}$ for some $1 \leq j \leq \ell$, x is not adjacent to y in $\Gamma'(Z_n)$. Therefore the subgraph induced by A_j is an independent set of $\Gamma'(Z_n)$.

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- Let $x \in A_{d_j}$ and $y \in A_{d_{j'}}$ for $1 \leq j \neq j' \leq \ell$. Then x is adjacent to y in $\Gamma'(Z_n)$ if and only if $d_j \nmid d_{j'}$ and $d'_j \nmid d_j$

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- Subgraph induced by $A_j \cup A_{j'}$ is either a totally disconnected graph or $K_{|A_j|,|A_{j'}|}$ for all $1 \leq j \neq j' \leq k$

Theorem

Let $n \geq 2$ be an integer. Then the following conditions are equivalent:

(1). *The graph $\Gamma'(Z_n)$ is well-covered.*

(2). *The graph $\Gamma'(Z_n)$ is Cohen–Macaulay.*

(3). *The graph $\Gamma'(Z_n)$ is well-covered vertex-decomposable.*

(4). *The graph $\Gamma'(Z_n)$ is Gorenstein.*

(5). $\Gamma'(Z_n)$ *is a power of a prime.*

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