COHEN–MACAULAYNESS OF A CLASS OF CIRCULANT GRAPHS

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Outline

Introduction

- Algebraic combinatorics is a branch of mathematics that applies techniques from abstract algebra to combinatorial problems and vice versa. Among its rapidly evolving subfields, combinatorial commutative algebra has gained significant attention in recent decades, emerging as a dynamic area of mathematical research.

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- In which commutative ring theory was born in the late 19th century with the works of famous authors like D. Hilbert, E. Noether, F.S Macaulay and W. Krull.

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- Recently, the combinatorial methods have been employed for studying these rings.

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Also, a Noetherian ring is said to be Cohen–Macaulay if all of its localizations at all maximal ideals are Cohen–Macaulay.

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- *Let G be a simple graph whose vertex set $V(G) = \{x_1, x_2, \ldots, x_n\}$ and $K[x_1, x_2, \ldots, x_n]$ be a polynomial ring on n variables over a field K. The edge ideal of G, denoted by $I(G)$, is the ideal generated by all the edges of G in $K[x_1, x_2, \ldots, x_n]$. That is $I(G) = \langle x_i x_j | \{x_i, x_j\} \in E(G) \rangle \subset K[x_1, x_2, \ldots, x_n]$.*

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- A graph is Cohen–Macaulay iff its edge ring is Cohen–Macaulay.

Simplicial Complex

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- Richard P. Stanley is well known for his fundamental and important contributions to combinatorics and its relationship to algebra and geometry, in particular in the theory of simplicial complexes.
- To each simpicial complex Δ, Stanley associated a quotient ring $K[\Delta]$, called Stanley-Reisner ring (or face ring) of Δ in such a manner that the combinatorial properties of the simpicial complex Δ are intimately related with the algebraic properties of the Stanley-Reisner ring $K[\Delta]$.

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- Each element $F \in \Delta$ is referred to as a *face* of Δ.
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- A *facet* of Δ is a face that is maximal with respect to inclusion.
- The simplicial complex Δ is said to be *pure* if all its facets have the same cardinality.

Definition

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K. The Stanley–Reisner ideal I_Δ defined as
$I_\Delta = \langle (x_{i_1} \cdot x_{i_2} \cdots x_{i_r}) \mid i_1 < i_2 \ldots < i_r, \{x_{i_1}, x_{i_2}, \ldots x_{i_r}\} \notin \Delta \rangle$,
and its Stanley-Reisner ring of Δ, denoted by $K[\Delta]$, is defined to be $K[\Delta] = R/I_\Delta$.

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Definition (Well-covered Graphs)

If all maximal independent sets of G have the same size, then G is called well-covered.

- Every Cohen–Macaulay graph is also well-covered.

- For a graph G, the *independence complex* of G, denoted by $\Delta(G)$, is a simplicial complex whose faces represent the independent sets of G, and whose facets correspond to the maximal independent sets of G.

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Definition

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K. The Stanley–Reisner ideal $I_{\Delta(G)}$ defined as

$$I_{\Delta(G)} = \langle x_i x_j | \{x_i, x_j\} \in E(G) \rangle = I(G)$$

and its Stanley-Reisner ring of $\Delta(G)$ is the edge ring

$$R/I_{\Delta(G)} = R/I(G).$$

A graph G is called *Cohen–Macaulay* (respectively *well-covered*, *shellable*) if its independence complex $\Delta(G)$ satisfies the corresponding property, that is $\Delta(G)$ is Cohen–Macaulay (respectively pure, shellable).

Definition

A simplicial complex Δ of dimension d is said to be pure shellable if it is pure and its facets (i.e., maximal faces) can be arranged in an order $F_1, \ldots, F_s$ such that

$$\overline{F}_i \cap \left(\bigcup_{j=1}^{i-1} \overline{F}_j \right)$$

is a pure $d-1$- dimensional complex for every $i \geq 2$. Here $\overline{F}_i = \{\sigma \in \Delta \mid \sigma \subset F_i\}$.

If Δ is pure shellable, $F_1, \ldots, F_s$ is called a *shelling*.

Proposition (1, Theorem 6.3.23)

Let Δ be a simplicial complex. If Δ is pure shellable, then Δ is Cohen–Macaulay over any field K.

[1] Villarreal, R.H.: Monomial Algebras. Monographs and Research Notes in Mathematics, 2nd edn. CRC Press, Boca Raton (2015)

Cayley Graph

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- Cayley graphs are now regarded as a fundamental tool in *group theory*, *algebraic graph theory*, and *combinatorics*.

Definition (Cayley Graph)

Let G be a group with identity element e and S a nonempty subset of G such that $e \notin S$ and $g^{-1} \in S$ for every $g \in S$. The Cayley graph $\mathrm{Cay}(G, S)$ is the simple graph with vertex set G, and two distinct vertices g and h are adjacent if and only if $g^{-1}h \in S$.

Definition (Circulant graph)

A circulant graph is the Cayley graph $\mathrm{Cay}(\mathbb{Z}_n^+, S)$ *for some* $S \subseteq \mathbb{Z}_n \setminus \{0\}$ *such that* $S = \{-x \mid x \in S\}$*.*

Circulant Graph

- In 2011, Brown et al., applied the theory of independence polynomials to show that several families of circulants are indeed well-covered. Later, 1n 2014, Vander Meulen et al. characterized its Cohen–Macaulayness.

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- In 2021, Hoang et al., characterized when certain classes of circulant graphs are well-covered, Cohen–Macaulay, Buchsbaum or Gorenstein.

Circulant Graph

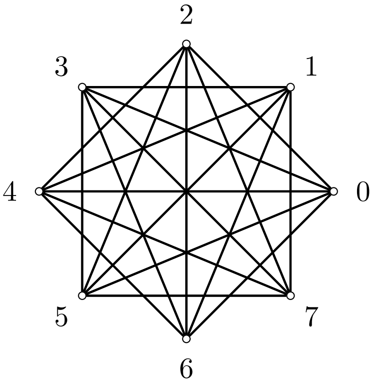
Definition

For each integers $n \geq 3$ and $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$, define the set

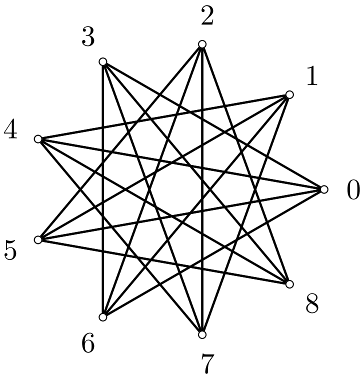
$$S_k = \{x \in [n] : \left\lfloor \frac{n-1}{2} \right\rfloor - (k-1) \leq x \leq \left\lceil \frac{n-1}{2} \right\rceil + k\}.$$

The circulant graph $Cir_n(S_k)$ is a simple graph with generating set S_k, whose vertex set is $[n] = \{0, \ldots, n-1\}$, where two distinct vertices x and y are adjacent if and only if either $|x-y| \in S_k$ or $n - |x-y| \in S_k$.

Examples



$Cir_8(S_2)$



$Cir_9(S_2)$

Results

Lemma

Let $n \geq 3$ and $1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ be two integers. Then for each $x \in [n]$, the set $I = \{x, x+1, \ldots, x + \left\lfloor \frac{n-1}{2} \right\rfloor - k\}$ is a maximal independent set of $Cir_n(S_k)$.

Lemma

Let $n \geq 3$ *and* $1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ *be integers. Then the independence number* $\alpha(Cir_n(S_k)) = \left\lfloor \frac{n-1}{2} \right\rfloor - k + 1$.

Main Theorem

Theorem

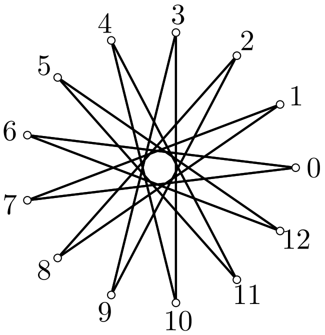
Let $n \geq 3$ and $1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ be integers. Then following conditions are equivalent:

(i). *The graph $Cir_n(S_k)$ is well-covered,*

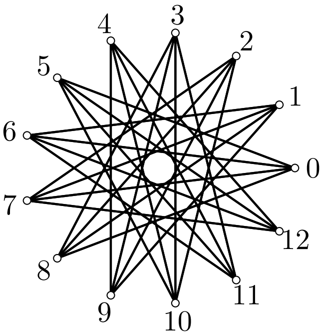
(ii). *The graph $Cir_n(S_k)$ is Cohen–Macaulay,*

(iii).

$$k \geq \begin{cases} \frac{n-1}{6} & \text{if } n \text{ is odd} \\ \frac{n-4}{6} & \text{if } n \text{ is even.} \end{cases}$$



$Cir_{13}(S_1)$
Non Cohen–Macaulay



$Cir_{13}(S_2)$
Cohen–Macaulay

- Since $Cir_{13}(S_1)$ is non Cohen–Macaulay, corresponding edge ring $K[x_0, x_1, \ldots, x_{12}]/\big(x_0x_6, x_0x_7, x_1x_7, x_1x_8, \ldots, x_5x_{11}, x_5x_{12}\big)$ is non Cohen-Macaulay.
- Since $Cir_{13}(S_2)$ is Cohen–Macaulay, corresponding edge ring $K[x_0, x_1, \ldots, x_{12}]/\big(x_0x_5, x_0x_6, x_0x_7, x_0x_8, x_1x_6, \ldots, x_4x_{11}, x_4x_{12}\big)$ is Cohen–Macaulay.

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