The Zero-Divisor Graph of a Commutative Ring

Dr. T. Asir

Associate Professor

Department of Mathematics

Pondicherry University

Puducherry 605 014, India

A MINI WORKSHOP ON GRAPHS, RINGS, AND MODULES
22 October, 2025

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The idea of constructing a graph from a algebraic structure was originated by Cayley in 1878.

Notations

The following notations are followed in this presentation.

$(R, +, .)$	Commutative ring
0	Additive identity
1	Multiplicative identity
(x)	Ideal generated by an element $x \in R$

Basic concepts

Definition

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(i) $(I, +)$ is a subgroup of $(R, +)$.

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A non-zero element $x \in R$ is said to be regular if x is not a zero-divisor in R. The set of all regular elements is denoted by $Reg(R)$. i.e., $Reg(R) = R - Z(R)$.

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Definition

The *zero-divisor graph* of R, denoted by $\Gamma(R)$, as the simple graph whose vertex set is $Z(R)^*$ and such that two distinct vertices $x, y \in Z(R)^*$ are adjacent if $xy = 0$.

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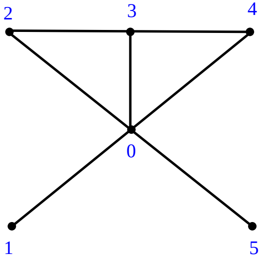
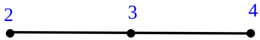


I. Beck



D. F. Anderson

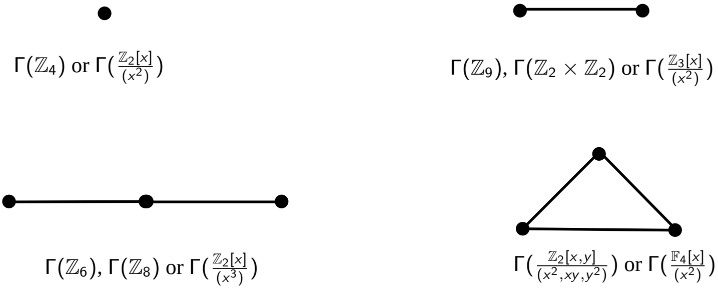
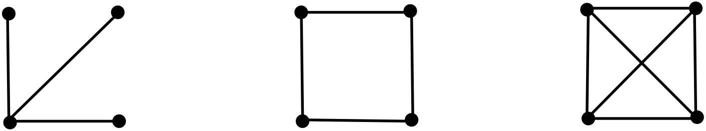


Figure: Zero-divisor graphs of rings of order less than three



$\Gamma(\mathbb{Z}_2 \times \mathbb{F}_4)$ $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)$ $\Gamma(\mathbb{Z}_{25})$ or $\Gamma(\frac{\mathbb{Z}_5[x]}{(x^2)})$

Figure: Zero-divisor graphs of rings of order four

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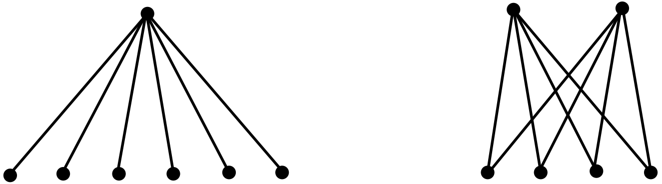
- Similarly $b + d = c$. Hence $b = a + c = a + b + d$; so $a + d = 0$. Thus $bd = b(-a) = 0$, a contradiction.



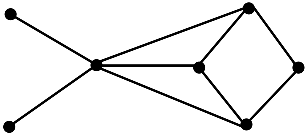
$\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_7)$ $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_5)$

Figure: Complete bipartite zero-divisor graphs

(a). $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4)$ and $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2))$

(b). $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

Figure: Zero-divisor graphs on 5 and 6 vertices



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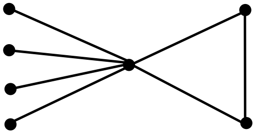
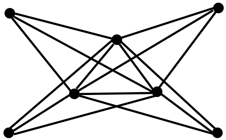


(b). $\Gamma(\mathbb{Z}_{16})$, $\Gamma(\mathbb{Z}_2[x]/(x^4))$, $\Gamma(\mathbb{Z}_4[x]/(x^2+2))$,

$\Gamma(\mathbb{Z}_4[x]/(x^2+2x+2))$ and $\Gamma(\mathbb{Z}_4[x]/(x^3-2, 2x^2, 2x))$

Figure: Zero-divisor graphs on 7 vertices



(a). $\Gamma(\mathbb{Z}_2[x,y]/(x^3,xy,y^2))$, $\Gamma(\mathbb{Z}_8[x]/(2x,x^2))$,

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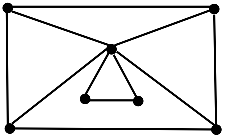

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- For any $r, s \in J$, $(r - s)y = 0$, so $ann(y) \subset Z(R)$ is infinite, a contradiction.

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Moreover, Zero divisor graphs were defined and studied for non-commutative rings, near rings, semi-groups, modules, lattices and posets.

The total graph of a commutative ring

In 2008, D. F. Anderson and A. Badwai introduced a new graph called the total graph of a commutative ring.

Definition

The *total graph* of R, denoted by $T_\Gamma(R)$, is the undirected simple graph with all the elements of R as vertices and for distinct vertices x and y are adjacent if $x + y \in Z(R)$. The two induced subgraphs with vertex set $Z(R)$ and $Reg(R)$ are denoted by $Z_\Gamma(R)$ and $Reg_\Gamma(R)$ respectively.

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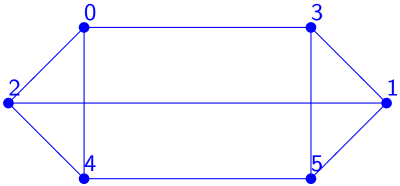

$T_\Gamma(\mathbb{Z}_6)$

Example

The total graph of a local ring $\mathbb{Z}_9$.

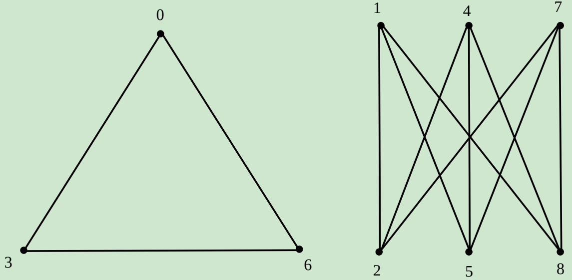

Figure 3.2: $T_\Gamma(\mathbb{Z}_9)$

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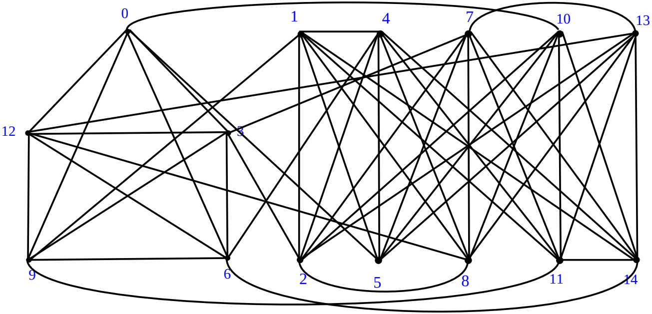

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$$deg(v) = \begin{cases} |Z(R)| - 1 & if\ 2v \in Z(R) \\ |Z(R)| & if\ 2v \notin Z(R). \end{cases}$$

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(ii) *If $2 \notin Z(R)$, then $deg(v) = |Z(R)| - 1$ for every $v \in Z(R)$ and $deg(v) = |Z(R)|$ for every vertex $v \notin Z(R)$.*

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When $T_\Gamma(R)$ is not connected?

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Theorem

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Thus $T_\Gamma(R)$ is disconnected whenever $Z(R)$ is an ideal of R.

Structure theorem of $T_\Gamma(R)$ when $Z(R)$ is an ideal of R.

Theorem

[D. F. Anderson et al.] (J. Algebra) *Let R be a commutative ring such that $Z(R)$ is an ideal of R, and let $|Z(R)| = \lambda$ and $|R/Z(R)| = \mu$.*

$$T_\Gamma(R) = \begin{cases} K_\lambda \cup \underbrace{K_\lambda \cup K_\lambda \cup \ldots \cup K_\lambda}_{(\mu-1)\ \textit{copies}} & \textit{if } 2 \in Z(R) \\ K_\lambda \cup \underbrace{K_{\lambda,\lambda} \cup K_{\lambda,\lambda} \cup \ldots \cup K_{\lambda,\lambda}}_{(\frac{\mu-1}{2})\ \textit{copies}} & \textit{if } 2 \notin Z(R). \end{cases}$$

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Thus $Reg_\Gamma(R)$ is the union of $\mu - 1$ disjoint complete subgraphs.

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Thus $T_\Gamma(R)$ is connected whenever $Reg_\Gamma(R)$ is connected.

Theorem

[D. F. Anderson et al.] (J. Algebra) *Let R be a commutative ring and $Z(R)$ is not an ideal of R. Then $T_\Gamma(R)$ is connected if and only if $(Z(R)) = R$.*

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Thus $0 \to b_1 \to b_2 \cdots \to b_n = x$ is a path from 0 to x in $T_\Gamma(R)$.

Hence there is a path between any two vertex via 0.

Corollary

If R is a finite commutative ring such that $Z(R)$ is not an ideal of R, then $T_\Gamma(R)$ is connected.

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Note that every regular element of a finite commutative ring is a unit.

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Note that every regular element of a finite commutative ring is a unit.

Thus $R = (x, y)$.

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QUESTIONS/QUERIES

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THANK YOU