Leavitt Path Algebras of Graphs Defined on Groups

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Directed Graphs

A directed graph E consists of a set of vertices E_0, a set of edges E_1, and maps $s, r : E_1 \to E_0$. For each $e \in E_1$, $s(e)$ is the *source* and $r(e)$ is the *range/target* of e

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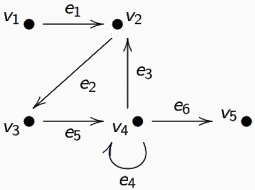
$$s(e) \xrightarrow{e} r(e)$$

If $r(e) = s(e)$, the edge e is called a **loop**.

Definition For a vertex v:

- if $s^{-1}(v) = \emptyset$, then v is a **sink**;
- if $r^{-1}(v) = \emptyset$, then v is a **source**;
- if $0 < |s^{-1}(v)| < \infty$, then v is a **regular vertex**.



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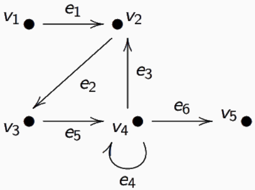

Definition A directed graph is **row-finite** if all non-sink vertices are regular.

Definition The **adjacency matrix** of a directed graph E is the matrix $A_E = [a_{uv}] \in \mathbb{Z}^{E_0} \times \mathbb{Z}^{E_0}$, where

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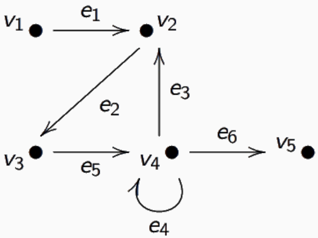

E

$$A_E = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Some Families of Directed Graphs on Groups

Let G be a finite group.

Cayley graph

Let S be a generating set of G. The **Cayley graph** of G with respect to S is the directed graph with vertex set G and with a directed edge from g to h iff there exists $s \in S$ such that $h = gs$. We denote it by $\Gamma(G, S)$.

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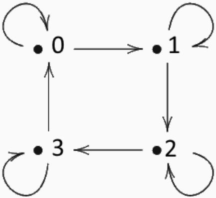
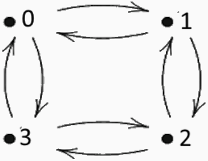


$\Gamma(\mathbb{Z}_4, \{0, 1\})$



$\Gamma(\mathbb{Z}_4, \{1, 3\})$

(Directed) Power Graph

The **(directed) power graph** of G, denoted $Pow(G)$, is the directed graph with vertex set G and a directed edge from g to h with $g \neq h$ if and only if there exists an integer m such that $h = g^m$.

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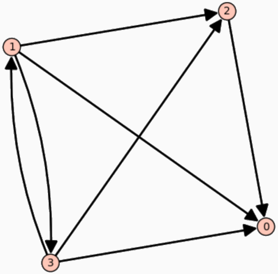
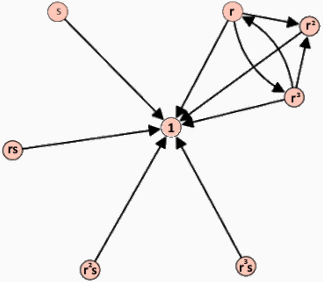


$Pow(\mathbb{Z}_4)$



$Pow(D_8)$

$D_8 = \langle r, s \mid r^4 = s^2 = 1,\ srs = r^3 \rangle$

(Directed) Punctured Power Graph

In $Pow(G)$, the vertex corresponding to the identity element is a sink. Deleting this vertex yields the **(directed) punctured power graph** of G, denoted $Pow^*(G)$.

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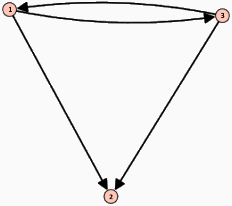

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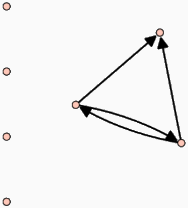

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Let E be a directed graph and K a field.

Definition The Leavitt path algebra $L_K(E)$ of E over K is the K-algebra generated by E_0, E_1, and $E_1^* = \{e^* \mid e \in E_1\}$ subject to:

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If $|E_0| < \infty$, then the sum of all vertices is the identity of $L_K(E)$.

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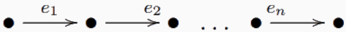

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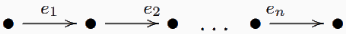

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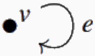

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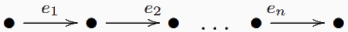

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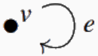

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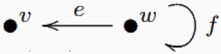


$\rightsquigarrow L_K(E) \cong K[x, y \mid yx = 1]$

Let $p = e_1 \cdots e_n$ be a path. If for some $1 \leq i \leq n$ there exists an edge $e \in E_1$ with $s(e) = s(e_i)$ and $e \neq e_i$, then e is an **exit** for p.

Let E be a finite graph with no cycles having exits.

Let $v_1, \ldots, v_n$ be the sinks of E.

Let $c_1, \ldots, c_m$ be the cycles of E.

Let w_i be a fixed vertex on c_i.

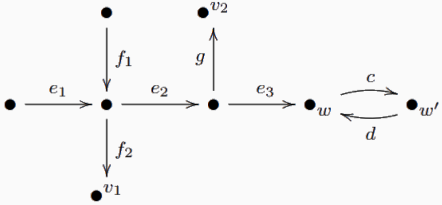
Let p_{ij} be the paths ending at v_i for $1 \leq j \leq k_i$, and

Let q_{ij} be the paths ending at w_i that do not traverse c_i for $1 \leq j \leq l_i$.

Then

$$L_K(E) \cong \bigoplus_{i=1}^{n} \mathbb{M}_{k_i}(K) \ \oplus \ \bigoplus_{i=1}^{m} \mathbb{M}_{l_i}\big(K[x, x^{-1}]\big) .$$

Example



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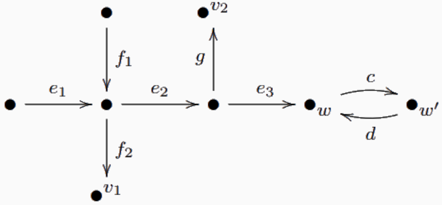

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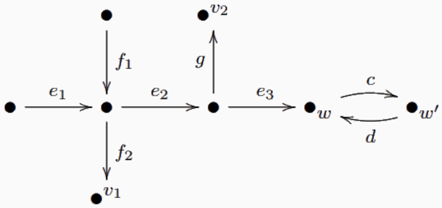

The graph E has two sinks v_1, v_2 and one cycle cd.

$$L_K(E) \cong \mathbb{M}_4(K) \ \oplus \ \mathbb{M}_5(K) \ \oplus \ \mathbb{M}_6\big(K[x, x^{-1}]\big)\,.$$

$\rightsquigarrow e^*e = f^*f = 1$ and $ee^* + ff^* = 1$

E

In this case, the maps $x \mapsto (e^*x, f^*x)$ and $(x, y) \mapsto ex + fy$ yield the isomorphism

$$L_K(E) \cong L_K(E) \oplus L_K(E).$$

Leavitt Algebras

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Suppose R is not IBN. Let $m \in \mathbb{N}$ be minimal with the property that $R^m \cong R^{m'}$ as left R-modules for some $m' > m$. For this m, let n denote the minimal such m'. In this case we say that R has **module type** (m, n).

Leavitt proved that for each $m < n$ there exist a unital K-algebra of type (m, n).

Theorem [Nam–Puc (2019) and Kanuni–Özaydın (2019)]
Let E be a directed graph with vertex set $E_0 = \{v_1, \dots, v_h\}$ whose regular vertices are $v_1, \dots, v_z$, and set

$$J_E = \begin{bmatrix} \mathrm{I}_z & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{M}_h(\mathbb{N}), \quad b = [1, \dots, 1]^t \in \mathbb{M}_{h\times 1}(\mathbb{N}).$$

Then $L_K(E)$ has the IBN property iff

$$\mathrm{rank}(A_E^t - J_E) < \mathrm{rank}\big([A_E^t - J_E \;\; b]\big).$$

Theorem [Nam–Puc (2019)]

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Theorem [–,Şimşek (2025)]

1) $L_K(Pow(\mathbb{Z}_n))$ has IBN for any $n \neq 10$.
2) $L_K(Pow^*(\mathbb{Z}_{p^n}))$ has IBN if and only if $p = 2$ and $n > 1$ or $p = 3$.

- $L_K(Pow(\mathbb{Z}_2^r)) \cong \mathbb{M}_{2^r}(K)$
- $L_K(Pow^*(\mathbb{Z}_2^r)) \cong K^{2^r-1}$
- $L_K(Pow^*(\mathbb{Z}_3^r)) \cong \bigoplus_{\frac{3^r-1}{2}} \mathbb{M}_2\big(K[x, x^{-1}]\big)$
- $L_K(Pow(\mathbb{Z}_3)) \cong L_K(Pow^*(\mathbb{Z}_4)) \cong ?$

Grothendieck Groups of Leavitt Path Algebras

Let $\mathcal{V}(R)$ be the set of isomorphism classes of finitely generated projective R-modules. For any two isomorphism classes $[P]$ and $[Q]$, we define

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Under this operation, $\mathcal{V}(R)$ forms an abelian monoid.

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$K_0(R) := K_0(\mathcal{V}(R))$

Let A_{ns} be the matrix obtained from the adjacency matrix A_E by deleting the rows corresponding to sinks of E.
Let I_{ns} be the matrix obtained from the identity $\mathrm{I} \in \mathbb{Z}^{E_0} \times \mathbb{Z}^{E_0}$ by deleting the rows corresponding to sinks of E.

Theorem [Abrams et al. (2005)] If E is row-finite, then

$$K_0(L_K(E)) \cong \mathrm{Coker}\Big((\mathrm{I}_{ns} - A_{ns})^{tr} : \bigoplus_{v \in \mathrm{Reg}(E)} \mathbb{Z} \to \bigoplus_{v \in E^0} \mathbb{Z}\Big).$$

Let E be a finite graph with n vertices.

Let S be the Smith normal form of $(\mathrm{I}_{ns} - A_{ns})^{tr}$ with rank r.

Let $s_1, \ldots, s_r$ be the nonzero diagonal entries so that $S = \mathrm{diag}(s_1, \ldots, s_r, 0,$

(Linear algebra: $s_1 \cdots s_k$ equals the gcd of all $k \times k$ minors of $(\mathrm{I}_{ns} - A_{ns})^{tr}$.)

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Corollary

$$K_0(L_K(E)) \cong \mathbb{Z}^{n-r} \oplus \mathbb{Z}/s_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/s_r\mathbb{Z}.$$

Theorem [Abrams, Ericson, Canto (2018)] Computed $K_0(L_K(\Gamma(\mathbb{Z}_n, \{1, 2\})))$ and $K_0(L_K(\Gamma(\mathbb{Z}_n, \{1, 3\})))$.

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Theorem [Das, Sen, Maity (2023)] For primes $p \geq 5$,

$$K_0(L_K(Pow^*(\mathbb{Z}_p))) \cong \underbrace{\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2}_{(p-3)} \oplus \mathbb{Z}_{2p-6}.$$

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Corollary For a prime $p \geq 5$ and $m = \dfrac{p^n - 1}{p - 1}$,

$$K_0\big(L_K\big(Pow^*(\mathbb{Z}_p^n)\big)\big) \cong \underbrace{\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2}_{m(p-3)} \oplus \underbrace{\mathbb{Z}_{2p-6} \oplus \cdots \oplus \mathbb{Z}_{2p-6}}_{m}.$$

K_0 of $L_K\big(Pow^*(\mathbb{Z}_{p^n})\big)$

Assume $p \geq 3$.

$S_k =$ the circulant matrix of the vector $(0, 1, \ldots, 1) \in \mathbb{Z}^k$.

$B_k =$ the circulant matrix of the vector $(1, -1, \ldots, -1) \in \mathbb{Z}^k$.

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$$A_{Pow^*(\mathbb{Z}_{p^n})} = \begin{pmatrix} S_{p-1} & 0 & 0 & \cdots & 0 \\ 1 & S_{p^2-p} & 0 & \cdots & 0 \\ 1 & 1 & S_{p^3-p^2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 1 & 1 & 1 & \cdots & S_{p^n-p^{n-1}} \end{pmatrix}$$

The matrix $\mathrm{I} - A^{tr}_{Pow^*(\mathbb{Z}_{p^n})}$:

$$\begin{pmatrix} B_{p-1} & -1 & -1 & \cdots & -1 \\ 0 & B_{p^2-p} & -1 & \cdots & -1 \\ 0 & 0 & B_{p^3-p^2} & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & -1 \\ 0 & 0 & 0 & \cdots & B_{p^n-p^{n-1}} \end{pmatrix}$$

$C_{m+1}(x)$ is obtained from B_m by inserting the row $(-1, \ldots, -1) \in \mathbb{Z}^m$ just before the x-th row.
$D_m(x) = C_{m+1}(x)^t$.
$A_{m+1}(x, y)$ is obtained from $C_{m+1}(x)$ by inserting the column $(-1, \ldots, -1)^t \in \mathbb{Z}^{m+1}$ just before the y-th column.

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Here $\det B_k = (2 - k) \cdot 2^{k-1}$ and $\det A_k(x, y) = \pm 2^{k-1}$.

$$A_4(2,3) = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}, \quad C_4(2) = \begin{pmatrix} 1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

$$D_3(2) = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}.$$

Preliminary Result: The nonsingular $(k \times k)$ minors of $\mathrm{I} - A^{tr}_{Pow^*(\mathbb{Z}_{p^n})}$ have the form

$$\begin{pmatrix} N_{k_1} & -1 & -1 & \cdots & -1 \\ 0 & N_{k_2} & -1 & \cdots & -1 \\ 0 & 0 & N_{k_3} & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & -1 \\ 0 & 0 & 0 & \cdots & N_{k_s} \end{pmatrix}$$

where $N_{k_i} \in \{A_{k_i}(x_i, y_i), B_{k_i}, C_{k_i}(x_i), D_{k_i}(x_i)\}$, with $1 \leq i \leq s$, $1 \leq k_i \leq p^{n-s+i} - p^{n-s+i-1}$, and $k_1 + \cdots + k_s = k$.

Theorem The greatest common divisor of the determinants of the $(k \times k)$ minors of $\mathrm{I} - A^{tr}_{Pow^*(\mathbb{Z}_{p^n})}$ is

$$= \begin{cases} 1, & 1 \le k \le n, \\ 2^{k-n}, & n+1 \le k \le p^n - n - 1, \\ 2^{2k-p^n+1}, & p^n - n \le k \le p^n - 2, \\ \left| \prod_{i=1}^{n} (2 - \varphi(p^i))\, 2^{\varphi(p^i)-1} \right|, & k = p^n - 1 \text{ and } p \ge 5, \\ 0, & k = p^n - 1 \text{ and } p = 3. \end{cases}$$

For a prime p, set

$$\eta = \frac{\prod_{i=1}^{n}(2 - \varphi(p^i))\, 2^{\varphi(p^i)-1}}{2^{p^n-3}}.$$

Theorem[–,Kanuni Er, Șimșek (2025)]

$$K_0(L_K(Pow^*(\mathbb{Z}_{p^n}))) \cong \begin{cases} \underbrace{\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2}_{(3^n-2n-1)} \oplus \underbrace{\mathbb{Z}_4 \oplus \cdots \oplus \mathbb{Z}_4}_{(n-1)} \oplus \mathbb{Z}, & p = 3 \\ \underbrace{\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2}_{(p^n-2n-1)} \oplus \underbrace{\mathbb{Z}_4 \oplus \cdots \oplus \mathbb{Z}_4}_{(n-1)} \oplus \mathbb{Z}_{|\eta|}, & p \geq 5 \end{cases}$$

Theorem[–,Kanuni Er, Şimşek (2025)]

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Theorem[–, Şimşek (2025)]

$$K_0(L_K(Pow(\mathbb{Z}_{p^n}))) \cong \underbrace{\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2}_{p^n-2n \text{ times}} \oplus \underbrace{\mathbb{Z}_4 \oplus \cdots \oplus \mathbb{Z}_4}_{n-1 \text{ times}}.$$

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