On Subinjectivity Domains of Finitely Generated Modules

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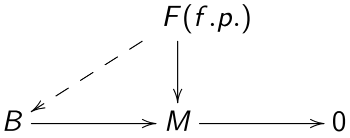
- M is **finitely presented** if $\exists$ an exact sequence $R^m \to R^n \to M \to 0$.
- Example: 1) $\mathbb{Q}_{\mathbb{Z}}$ is flat (=torsion-free). 2) Every direct summand (so every subspace of a vector space) is pure. 3) $\mathbb{Z}/n\mathbb{Z}$ is f.p.

f-projective modules

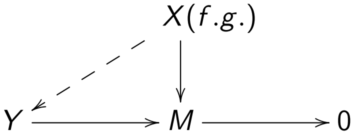
A characterzation of flat modules:

M is flat $\Leftrightarrow$ every s.e.s $\mathbf{E}: 0 \to A \to B \to M \to 0$ is pure-exact $\Leftrightarrow$ every f. p. module F is projective w.r.t $\mathbf{E}$:



f-projective modules

1. An epim. $f : A \to B$ is **finitely split** (or $\ker f$ is fin. split in A) if every f.g. module is projective w.r.t. f; and M is **finitely projective** (or f-projective) if every epim. onto M is finitely split:



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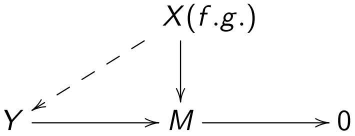
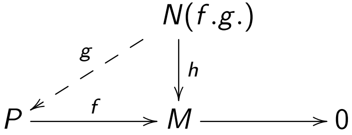


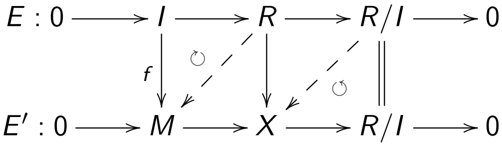
2. Equivalently, every homom. $h : N \to M$ with a f.g. module N factors through a projective module P:



$$f \circ g = h$$

dual of f-projective modules

- As a dual of (1): If M were defined to be **f-injective** provided that M is fin. split in every extension, i.e., every f.g. module is projective w.r.t any s.e.s. starting with M, then M would be injective by Baer's criterion. If $f : I \to M$, then



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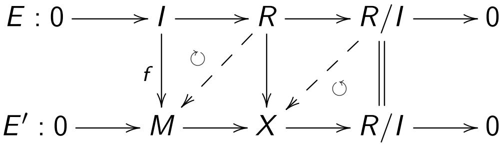

- So, we consider a dual of (2), and define FG-injective modules, as a generalization of injective modules.

FG-injective modules

- M is **FG-injective** if every homom. $f : M \to N$ with a f.g. module N factors through an injective module E:

$$\begin{array}{ccc} M & \xrightarrow{f} & N(f.g.) \\ {\scriptstyle h}\downarrow & \nearrow {\scriptstyle g} & \\ E & & \end{array} \qquad f = g \circ h$$

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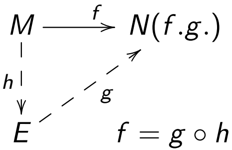

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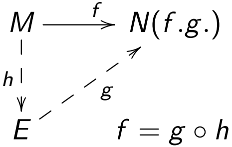

- Every injective module is FG-injective, but the converse is not true.
- Every f.g. FG-injective module is injective.

Some facts about FG-injective modules

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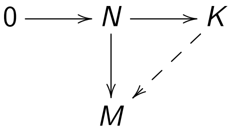
- R is a right self-injective iff every projective R-module is FG-injective.
- **Exm.** Let $R = \prod_{i=1}^{\infty} K$ (K a field). Then R is self-injective, but not noetherian, and so not a QF-ring. So, there is projective module P that is not injective. Therefore, $M = \oplus_{i=1}^{\infty} P$ is projective, and so FG-injective, but not an injective R-module.

subinjectivity domain (Aydogdu-Lopez [2])

- M is N-**subinjective** if M is injective w.r.t any s.e.s starting with N:



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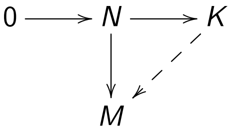

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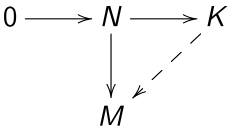

- The **subinjectivity domain** of M:
  $\underline{\mathfrak{In}}^{-1}(M) = \{X \mid M \text{ is X-subinjective}\}$.
- Clearly, M is injective iff $\underline{\mathfrak{In}}^{-1}(M) = \text{Mod-}R$.
- M is FG-injective iff every f.g. module N is M-subinjective, that is,

$$\mathcal{FGI} = \bigcap_{N \in \mathcal{FG}} \underline{\mathfrak{In}}^{-1}(N)$$

fg-indigent modules

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existence

If R is right noetherian, then fg-indigent modules exist. Indeed, $M = \prod_{X \in \Gamma} X$, where Γ is any complete set of repr. of fin. gen. (=fin. presented) modules, then

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so M is fg-indigent. For instance, over a comm. hereditary noetherian ring, fg-indigent=indigent.

subinjective portfolios (Holston et.al. [4])

- A *basic si-portfolio* $\mathcal{I}$ is defined as $\mathcal{I} = \underline{\mathfrak{In}}^{-1}(M) = \{N \in \text{Mod-}R \mid \text{Hom}(N, M) = 0\}$ (i.e. a torsion class of a torsion theory cogenerated by M).

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Theorem-1

Over a right noetherian ring R, $\mathcal{IN}$ ($\mathcal{FGI}$) is a basic si-portfolio iff every (f.g.) module is a direct sum of an injective module and a module containing no *h-divisible* submodules iff R is right hereditary.

subinjective profile

We define (right) subinjective profile of f.g. modules as

$$\mathfrak{siP}(\mathcal{FG}) := \{\underline{\mathfrak{In}}^{-1}(M) \mid M \in \mathcal{FG}\}.$$

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- $|\mathfrak{si}\mathfrak{P}(\mathcal{FG})| = 2$ iff every f.g. module is either (FG-)injective or fg-indigent (when R is not semisimple).

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- If R is not a right V-ring and $\mathfrak{si}\mathfrak{P}(\mathcal{FG})$ is a linearly ordered set, then there exist an fg-indigent module, and a unique simple non-injective module, up to iso., say W.

Note that h-disible R-modules are injective iff R is a Dedekind domain.

Theorem-3

If $\mathfrak{siP}(\mathcal{FG})$ is a linearly ordered set, then every f.g. h-divisible module is injective iff $\underline{\mathfrak{In}}^{-1}(N)$ is a basic si-portfolio for some f.g. module N.

Recall that R is right **SI-ring** ($Z_2(R_R)$**-semiperfect**) iff each singular (nonsingular) right R-module is injective. And, R is right semiartinian if every nonzero right R-module has a nonzero socle.

Theorem-4

- If each R-module is either injective or fg-indigent, and $R/J(R)$ is right noetherian, then R is a right semiartinian ring with W.

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- If W is projective, then $|\mathfrak{siP}(\mathcal{FG})| = 2$ iff R is a right hereditary, right perfect ring.
- If W is singular and R is not self-injective, then $|\mathfrak{siP}(\mathcal{FG})| = 2$ iff R is a right Artinian ring and $\mathcal{FGI} = \mathcal{NC}$.

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Thank you for your attention!