On the Proper Flat Profiles of Some Module Classes

Zübeyir TÜRKOĞLU
A MINI WORKSHOP ON GRAPHS, RINGS, AND MODULES
Department of Mathematics
Faculty of Science, Dokuz Eylül University
This is joint work with
Prof. Dr. Yılmaz DURĞUN and Prof. Dr. Salahattin ÖZDEMİR
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Outline

Vector Space V over a field $(\mathbf{k}, +, \cdot)$

Definition: A vector space over a field $(\mathbf{k}, +, \cdot)$ is a set V together with the operations of addition $\oplus : V \times V \longrightarrow V$ and a multiplication $\odot : \mathbf{k} \times V \longrightarrow V$ such that $(c, v) \longmapsto c \odot v = cv$ satisfying the fallowing properties;

1. $(V, \oplus)$ is an abelian group.
2. $c \odot (v_1 \oplus v_2) = c \odot v_1 \oplus c \odot v_2$ for all $c \in \mathbf{k}$ and for all $v_1, v_2 \in V$.
3. $(c_1 + c_2) \odot v = (c_1 \odot v) \oplus (c_2 \odot v)$ for all $c_1, c_2 \in \mathbf{k}$ and for all $v \in V$.
4. $(c_1 \cdot c_2) \odot v = c_1 \odot (c_2 \odot v)$ for all $c_1, c_2 \in \mathbf{k}$ and for all $v \in V$.
5. $1_{\mathbf{k}} \odot v = v$ for all $v \in V$.

A *right* R-module M over a ring $(R, +, \cdot)$

Definition: A *right* R-module over a ring $(R, +, \cdot)$ is a set M together with the operations of addition $\oplus : M \times M \longrightarrow M$ and a multiplication $\odot : M \times R \longrightarrow M$ such that $(m, r) \longmapsto m \odot r = mr$ satisfying the fallowing properties;

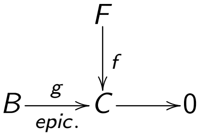
1. $(M, \oplus)$ is an abelian group.
2. $(m_1 \oplus m_2) \odot r = m_1 \odot r \oplus m_2 \odot r$ for all $r \in R$ and for all $m_1, m_2 \in M$.
3. $m \odot (r_1 + r_2) = (m \odot r_1) \oplus (m \odot r_2)$ for all $r_1, r_2 \in R$ and for all $m \in M$.
4. $m \odot (r_1 \cdot r_2) = (m \odot r_1) \odot r_2$ for all $r_1, r_2 \in R$ and for all $m \in M$.
5. $m \odot 1_R = m$ for all $m \in M$.

Main difference and definition of a free R-module

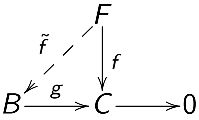
All vector spaces **have a basis** but **all** R-modules **may not have** a basis.
An R-module F is said to be **free** if it has a basis $\{x_i\}_{i \in I}$, that is, $\forall v \in F, \exists$ **unique** $(r_i)_{i \in I} \in \prod_{i \in I} R$ such that $r_i = 0$ for all but finitely many $i \in I$ and $v = \sum_{i \in I} r_i v_i$.

Free R-modules have an **important** property

For the given diagram



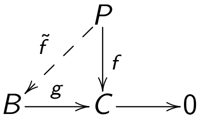
$\exists$ a R-module homomorphism $\tilde{f} : F \longrightarrow B$ s.t. the following diagram



commutes, that is, $g \circ \tilde{f} = f$.

Definition of a Projective R-module

Definition: An R-module P is said to be **projective** if $\forall$ epim. $g : B \longrightarrow C$ of R-modules and $\forall$ R-module homom. $f : P \longrightarrow C$, $\exists$ a R-module homom. $\tilde{f} : P \longrightarrow B$ s.t. $g \circ \tilde{f} = f$, i.e, for the given diagram



commutes.

Equivalent condition for projectivity by homologicaly

An R-module P is **projective** iff $\mathrm{Hom}_R(P, *)$ is an **exact** functor.

If $\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is a **short exact sequences** of R-modules, then $\mathrm{Hom}_R(P, \mathbb{E})$ is **also exact**, that is,

$0 \longrightarrow \mathrm{Hom}_R(P, A) \longrightarrow \mathrm{Hom}_R(P, B) \xrightarrow{g\circ -} \mathrm{Hom}_R(P, C) \longrightarrow 0$ is exact.

Injectivity is the dual concept of the projectivity

An R-module E is **injective** iff $\mathrm{Hom}_R(*, E)$ is an **exact** functor.

If $\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is a short exact sequence of R-modules, then $\mathrm{Hom}_R(\mathbb{E}, E)$ is also exact, that is,

$0 \longrightarrow \mathrm{Hom}_R(C, E) \longrightarrow \mathrm{Hom}_R(B, E) \xrightarrow{- \circ f} \mathrm{Hom}_R(A, E) \longrightarrow 0$ is exact.

Definition of a Flat **right** R-module

We call an R-module M is **flat** if $M \otimes_R *$ is an **exact** functor.

If $\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is a short exact sequence of left R-modules, then $M \otimes_R \mathbb{E}$ is also exact, that is,

$M \otimes_R \mathbb{E} : 0 \longrightarrow M \otimes_R A \longrightarrow M \otimes_R B \longrightarrow M \otimes_R C \longrightarrow 0$ is exact.

In an Other Words

Definitions are given **by using all short exact sequences** of R-modules, that is,

P is **projective** iff
$P \in \pi(\mathcal{A}bs) := \{A \mid A \textit{ is a R-module } \mathrm{Hom}_R(A, *) \textit{ is exact}\}$.

E is **injective** iff
$E \in \iota(\mathcal{A}bs) := \{A \mid A \textit{ is a R-module } \mathrm{Hom}_R(*, A) \textit{ is exact}\}$.

M is **flat** iff
$M \in \tau(\mathcal{A}bs^l) := \{A \mid A \textit{ is a R-module } A \otimes_R * \textit{ is exact}\}$.

Like a dual definition of $\pi(\mathcal{A})$, $\iota(\mathcal{A})$, $\tau(\mathcal{A})$

If $\mathbb{E}$ is a short exact sequence of R-modules, then
$\pi^{-1}(\mathcal{M}) := \{\mathbb{E} \mid \mathrm{Hom}_R(A, \mathbb{E}) \textit{ is exact for any } A \in \mathcal{M}\}$
$\iota^{-1}(\mathcal{M}) := \{\mathbb{E} \mid \mathrm{Hom}_R(\mathbb{E}, A) \textit{ is exact for any } A \in \mathcal{M}\}$
If $\mathbb{E}$ is a short exact sequence of **left** R-modules, then
$\tau^{-1}(\mathcal{M}) := \{\mathbb{E} \mid A \otimes_R \mathbb{E} \textit{ is exact for any } A \in \mathcal{M}\}$

Definition of Proper Class

A class of s.e.s $\mathcal{A}$ is said to be **proper** if it satisfies the following properties:

(P_1) $\mathcal{A}$ is closed under isomorphisms.

(P_2) $\mathcal{A}$ contains all splitting short exact sequences.

(P_3) (i) The composite of two $\mathcal{A}$-monomorphisms is an $\mathcal{A}$-monomorphism if this composite is defined.
(ii) The composite of two $\mathcal{A}$-epimorphisms is an $\mathcal{A}$-epimorphism if this composite is defined.

(P_4) (i) If g and f are monomorphisms, and $g \circ f$ is an $\mathcal{A}$-monomorphism, then f is an $\mathcal{A}$-monomorphism.
(ii) If g and f are epimorphisms, and $g \circ f$ is an $\mathcal{A}$-epimorphism, then g is an $\mathcal{A}$-epimorphism.

Examples of Proper Classes

- $\mathcal{S}plit$
- $\mathcal{A}bs$
-

$$\begin{aligned} \mathcal{P}ure &= \pi^{-1}(\{\text{all finitely presented } R\text{-modules}\}) \\ &= \tau^{-1}(\{\text{all finitely presented } \textit{left } R\text{-modules}\}) \\ &= \tau^{-1}(\{\text{all } \textit{left } R\text{-modules}\}) \end{aligned}$$

Some ring charac. using by inj., proj. and flatness

- semisimple Artinian ring.
- right (left) V-rings.
- von Neumann Regular Rings

Alternative perspectives of injectivity, projectivity and flatness

Let M and N be R-modules.

- M is called N-injective, $0 \to \bullet \to N \to \bullet \to 0$, $\mathfrak{In}^{-1}(M)$, poor module (see [Alahmadi et al., 2010], [Er et al., 2011], [Aydoğdu and Saraç, 2013]).

Alternative perspectives of injectivity, projectivity and flatness

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- M is called N-projective, $0 \to \bullet \to N \to \bullet \to 0$, $\mathfrak{Pr}^{-1}(M)$, p-poor (see [Holston et al., 2012]).
- M is called N-flat, $0 \to \bullet \to N \to \bullet \to 0$, $\mathcal{F}^{-1}(M)$, rugged (see [Büyükaşık et al., 2018]).
- M is called N-subinjective, $0 \to N \to \bullet \to \bullet \to 0$, $\underline{\mathfrak{In}}^{-1}(M)$, indigent (see [Aydoğdu and López-Permouth, 2011]).
- M is called N-subprojective, $0 \to \bullet \to \bullet \to N \to 0$, $\underline{\mathfrak{Pr}}^{-1}(M)$, p-indigent (see [Holston et al., 2015], [Durğun, 2015]).
- M is called N-subflat, $0 \to \bullet \to \bullet \to N \to 0$, $\mathcal{S}f^{-1}(M)$, f-test (see [Alizade and Durğun, 2017]).

The opposite of inj, proj. and flatness by Proper Classes

- An R-module M is called ι-indigent if $\iota^{-1}(M) = \mathcal{S}plit$. ([Aydoğdu and Durğun, 2023]).
- An R-module M is called π-indigent if $\pi^{-1}(M) = \mathcal{S}plit$. ([Durğun, 2024b]).
- An R-module M is called τ-rugged if $\tau^{-1}(M) = \mathcal{P}ure^{l}$. ([Durğun, 2024a]).

The right proper injective and projective profile of R

- The class $\iota(\text{Mod-}R) := \{\iota^{-1}(M) : M \in \text{Mod-}R\}$ which is called the (right) proper injective profile of a ring R. ([Aydoğdu and Durğun, 2023]).
- The class $\pi(\text{Mod-}R) := \{\pi^{-1}(M) : M \in \text{Mod-}R\}$ which is called the (right) proper projective profile of a ring R. ([Durğun, 2025]).

Some Results I

- The class $\tau(\mathcal{M}) := \{\tau^{-1}(M) : M \in \mathcal{M}\}$ which is called the (right) proper flat profile of a module class $\mathcal{M}$.

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- The class $\tau(\mathcal{M}) := \{\tau^{-1}(M) : M \in \mathcal{M}\}$ which is called the (right) proper flat profile of a module class $\mathcal{M}$.
- The following statements are equivalent for the ring R.
  1. $|\tau(\text{Mod-}R)| = 1$.
  2. Every right R-module is flat.
  3. Every right R-module is τ-rugged.
  4. $\tau(\mathcal{FP}) = \{\mathcal{P}ure^l\}$.
  5. $\tau(\mathcal{FP}) = \{\mathcal{A}bs^l\}$.
  6. R is a von Neumann regular ring.
- Let R be a right Noetherian ring. Then the following statements are equivalent.
  1. $|\tau(\text{Mod-}R)| = 2$.
  2. $\tau(\text{Mod-}R) = \{\mathcal{P}ure^l, \mathcal{A}bs^l\}$.
  3. $\tau(\mathcal{FP}) = \{\mathcal{P}ure^l, \mathcal{A}bs^l\}$.
  4. $\tau(\mathcal{FP}^l) = \{\mathcal{P}ure, \mathcal{A}bs\}$.
  5. $|\tau(\mathcal{FP})| = 2$.
  6. R has a unique singular, simple right R-module (up to isomorphism) and it is either Artinian serial ring with $J^2(R) = 0$ or right finitely Σ-CS, right SI ring.
- If R is a left Noetherian ring, then $|\tau(\text{Mod-}R)| \neq 3$.

Some Results II

- A ring R is of finite representation type if and only if $\tau(\text{Mod-}R) = \pi(R\text{-Mod})$ and $\tau(R\text{-Mod}) = \pi(\text{Mod-}R)$.
- Let R be a right pure-semisimple ring. $\tau(R\text{-Mod})$ is a chain if and only if R is an Artinian serial ring with unique (up to isomorphism) singular simple right R-module and $J^2(R) = 0$.

Some Results III

- Let R be an Artinian ring. If $\tau(\mathcal{FP})$ is a chain, then there is a ring direct sum $R \cong S \times T$, where S is a semisimple Artinian ring and T is an indecomposable ring which is either
  - (i) a right (left) n-saturated matrix ring over a local QF-ring, or
  - (ii) a hereditary Artinian serial ring with $J(T)^2 = 0$.
- Let R be a commutative Noetherian ring. If $\tau(\mathcal{FP})$ is a chain, then there is a ring direct sum $R \cong S \times T$, where S is semisimple Artinian ring and T is an indecomposable ring which is either Artinian uniserial ring with $J^2(T) = 0$ or hereditary Noetherian domain.
- Let R be a commutative Artinian local ring with the unique maximal ideal $\mathfrak{m} \neq 0$. Then $\tau(\mathcal{FP})$ is a chain if and only if R has a unique (up to isomorphism) singular simple right R-module, and it is a uniserial ring with $J^2(R) = 0$.

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